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COUNTER MODELS AND APPLICATIONS TO DETECTION PROBLEMS

BY
HANS ZWEIG

TECHNICAL REPORT NO. 87
April 12, 1963

PREPARED UNDER CONTRACT Nonr-225(52)
(NR-342-022)
FOR
OFFICE OF NAVAL RESEARCH



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Notation

λ	parameter of Poisson process
$\hat{\lambda}$	estimate of the parameter λ
T	time of operation of the detector on the stochastic process
T_1	time to the i^{th} event
X_1	a random variable denoting elapsed time between events ($i-1$) and i
t	a point in time, or an amount of elapsed time depending on the context
$F(t)$	inter-arrival time distribution of input stochastic process
$F_0(t)$	inter-arrival time distribution of output stochastic process
p	probability of an event
$N(T)$	number of (Poisson) input events to detector in $(0, T)$
$M(T)$	number of output events from detector in $(0, T)$
$E[\]$	expectation of $[\]$
$\text{var}[\]$	variance of $[\]$
$\varphi(s), (\varphi_0(s))$	Laplace transform of input (output) inter-arrival distribution
$\mu (\mu_0)$	mean of $F(t), (F_0(t))$
$\sigma^2 (\sigma_0^2)$	variance of $F(t), (F_0(t))$
ξ	asymptotic quantum efficiency of detector - the ratio of the asymptotic variance of the estimate of λ of the ideal detector to that of the detector model under consideration
τ	a time interval of fixed or variable duration following an input event to the detector
$G(\tau)$	the distribution function of τ
$[\]^*$	Laplace transform of $[\]$

τ_d a dead time interval following input event
 τ_c a coincidence interval following input event
 $S_j, j=1,2,\dots,R-1$ the states through which an R-fold coincidence counter must pass before registering an output event
 S_0 the ground state (the most stable state) of a coincidence counter and the state at $t = 0$
 $p_j(t, \tau)$ the probability density that state is j at time t , and has been for a time τ and that state R has not occurred in $(0, t)$ and state at $t = 0$ is S_0
 $\pi_j(s, \tau)$ Laplace transform of $p_j(t, \tau)$
 $p_j(t)$ probability that state at time t is j and that state R has not occurred in $(0, t)$
 $\mu(\tau)$ hazard function or conditional probability density of decay at time τ given survival to time τ
 $H_\lambda(T)$ probability that at least one count is registered in time interval $(0, t)$
 $H_\lambda(t) = H(\lambda T)$ for photographic detector without reciprocity failure (this defines absence of reciprocity failure)
 Q "fractional utilization factor" - an ambiguous notion discussed in text
 R number of photons required to make a photographic grain developable
 S the number of photographic grains involved in a photographic detection problem - S is large but the grains are confined to a small area
 $\mathcal{E} = \mathcal{E}(\lambda T, R)$ for a simple photographic detector without reciprocity failure
 $\mathcal{E} = \mathcal{E}(\lambda T, K(R)) \neq \sum K(R) \mathcal{E}(\lambda T, R)$ for a composite photographic detector without reciprocity failure
 $K(R)$ the distribution of R in the grains of a given photographic material

$\mathcal{E} = \mathcal{E}(\lambda, T, \tau_d, R) < \mathcal{E}(\lambda T, R)$ for a photographic material with high
intensity reciprocity failure [$\lambda T = \text{constant}$, λ large]

$\mathcal{E} = \mathcal{E}(\lambda, T, \tau_c, R) < \mathcal{E}(\lambda T, R)$ for a photographic material with low
intensity reciprocity failure [$\lambda T = \text{constant}$, λ small]

T' that amount of time in $(0, T)$ during which detector is not
"dead".

COUNTER MODELS AND APPLICATIONS TO DETECTION PROBLEMS

by

Hans Zweig

INTRODUCTION AND SUMMARY

Work in several aspects of detection theory has been in progress over the last two decades. Electronic engineers dealing with communication theory have looked into detection problems which develop from their studies. Another direction of work in detection theory stems from the work of physicists in the area of radiation and infrared detection. Detection theory is a term also applied in research problems and in other contexts. The concepts and methods of probability and statistics play important roles in these studies and vary with the character of the problems. Communication theory is most closely allied with the branch of statistics dealing with hypothesis testing and the work of transposing this statistical methodology into the field of communication engineering receives much attention. The second study mentioned above is concerned more with estimation problems and has not had such a straightforward development as the first. It has not been clear which physical and statistical concepts are useful nor how detectors (i.e., radiation detectors) should be evaluated. Here the major problem is one of determining how actual detector devices fall short of being ideal in some sense. In this case more than in communication theory there is a need to simulate mathematically or probabilistically the working of the detectors in order to understand and evaluate the detection process.

Radiation is often thought of as a Poisson stream of discrete quanta or events, or a combination of Poisson streams, and radiation detectors can therefore be thought of as estimators of the parameter or parameters of these streams. One criterion for judging the performance of radiation detectors which appears particularly useful is that of "quantum efficiency". This expression has been used in a variety of ways. Like most indices it has grown out of intuitive notions and thus some initial definitions appropriate for the original motivation must be used carefully in other settings. In Chapter I we give a specific definition which is used throughout the paper.

The history of this expression is interesting. Originally it was used simply to designate, for a given device, the ratio of the number of output events, M , to the number of input events, N ; a usage which can be shown to be quite inappropriate in a large number of situations (see Chapter IV). The modern version of this concept was first formulated by Rose (11) and later reintroduced under the designations "detective quantum efficiency" by Jones (5,6) and "quantum efficiency" by Fellgett (4). These writers defined the notion as a ratio of signal-to-noise-ratios of the "ideal detector" to that of the actual detector. Another definition was proposed by Zweig, Higgins, and MacAdam (20). It is the ratio, lying between zero and one, of variances of estimates of intensity (e.g. mean of a Poisson process) for an ideal detector (numerator) and for an actual device (denominator). They also demonstrated that this definition is equivalent to that used by the earlier writers when it is applied to their analyses. Using this concept of quantum efficiency, Zweig (18, 19) examined a simplified model of the

photographic process to study performance limitations of photographic detectors.

The purpose of the present work is to derive the quantum efficiency of a variety of detectors for which mathematical models already exist and to develop new models for some physical situations which can be examined in this light.

The plan of this work is as follows: In Chapter I, (a) a summary of the models which will be analyzed is given, (b) the applications of these models to specific detection problems is indicated, and (c) the methods of estimating the parameter of interest and the efficiency of the estimates is defined. Chapter II contains a summary of known dead time counter models and evaluates their quantum efficiency. In Chapter III a new class of counters, called coincidence counters, is introduced and evaluated. Chapter IV contains a discussion of photographic detectors. The dead time and coincidence counter models are used here to simulate the photographic effect known technically as "reciprocity failure", and the quantum efficiency of photographic detectors both with and without reciprocity failure is obtained. Chapter V contains two other applications of the counter models. In this chapter it is shown how combinations of counters can be used to solve more complex detection problems. One application involves using dead time and coincidence counters in series. This combination is useful in solving a radar ranging problem when the detection problem consists of determining the presence of a pulse pair in a Poisson noise stream. Another application involves using these counters in parallel. In this case the application of interest is that of determining the component parameters of a compound

Poisson stream. Even though many of these models are elementary in a physical sense their mathematical formulation and development require a fair amount of sophistication and manipulation.

CHAPTER I
MODELS, APPLICATIONS AND METHODS

1. Models

Various types of counter models have been introduced and discussed in the literature of the last twenty years. Type I and Type II counters have received the most extensive treatment. Type I counters are characterized by the fact that a registered event produces a dead time during which no further events can be registered. Type II counters are characterized by the fact that incoming events produce dead time during which further incoming events cannot be registered, although these events are capable of prolonging the dead time, that is, paralyzing the counter. Precise definitions of the counter models will be found in the sections in which these models are discussed and evaluated. Modifications of Type I and Type II counters have also been introduced and studied in the literature.

Although the problem of dealing with transformations of Poisson processes goes back at least as far as Bortkiewicz's paper of 1913 (1), the earliest formulation of what has come to be called the Type II counter seems to be that of Levert and Scheen in 1943 (7). The distinction between Type I and Type II counters was made somewhat earlier (1937) by Ruark and Brammer (12). Malmquist (8) considered the case of a Type I counter operating on a process with arbitrary inter-arrival time distribution $F(t)$, thus generalizing the Poisson input inter-arrival time distribution $F(t) = 1 - e^{-\lambda t}$ rather than the counter mechanism. Feller (3), confining himself to the classical Type I and Type II models, has

presented the use of straightforward probabilistic arguments and Laplace transforms to determine the Laplace transform of the inter-arrival time distribution of registered events. Takacs (16), using a similar approach, generalized the counter model so that each incoming event produces a dead time interval τ with probability p and another event occurring during this dead time, if one is created, will not be registered although it may also produce another dead time with the same probability. The case $p = 0$ produces the Type I counter, while $p = 1$ produces the Type II model. In this paper, we shall evaluate the quantum efficiency of Type I and Type II counters. These provide interesting comparisons with the coincidence counter models which will be developed, as well as with the photographic type of detectors.

Takacs' model is interesting in that it provides a bridge between the Type I and Type II counters. However, in doing this it also blurs the distinctive features of these counters and it is these that we want to exploit. Takacs has also dealt with a coincidence problem which is concerned with the probability of finding several dead time counters simultaneously locked. This is a different type of coincidence situation than the one we now present.

The simple coincidence counter model we now introduce is intended to work as follows: An incoming event, say a photon, is not directly recorded by the detector. Instead it changes the state of the detector say from S_0 to S_1 . The state S_1 is maintained for a fixed time following the incidence of the photon. If during this time interval no further arrival takes place, the detector reverts to state S_0 and no output count is recorded. If, however, while the detector is in state

S_1 another photon arrives an output count is registered and the detector immediately reverts to state S_0 . Thus for example if photons were incident on the detector at time t , $t + 2.0\tau$, $t + 2.1\tau$, $t + 2.5\tau$, and $t + 3\tau$ an output event would be registered at times $t + 2.1\tau$ and at $t + 3\tau$; or in this case five input events have produced two output events.

The R-fold coincidence counter is a generalization of the simple coincidence counter. Here, incoming events cause changes in the state of the detector from S_i to S_{i+1} provided the detector is in state S_i ($i = 0, 1, \dots, R-1$) at the time of the arrival of the input event. For $i = 1, 2, \dots, R-1$ the counter remains in state i for a time τ_i which is now taken to be a random variable with distribution independent of i . If no further event occurs during time τ_i following the last input the counter reverts to state S_0 without registering a count. Only if state S_R is reached is a count registered with the counter immediately reverting to S_0 .

The photographic detector is a counting device of an entirely different type but the counter models nevertheless play a part in the analysis of this type of detector. Basically the photographic detector (the emulsion or photographic plate) consists of an ensemble of many counters (the individual photographic grains), each of which is able to register a count only once. The fractional number of grains that develop is then a measure directly related to the average number of photons incident on each grain during the time of exposure. The simplest photographic detector consists of grains all of the same size and speed, i.e., each grain requires R photons to make it developable (that is, to register a count). Aside from variations in grain size

which will not be considered, actual photographic detectors consist of an ensemble of grains of varying speed so that R , the number of photons required for registering an output event, is a random variable with distribution dependent on the particular photographic material.

2. Applications

There is, a photographic effect which can be explained in terms of the dead time and coincidence counter models introduced above. This effect is known technically as reciprocity failure and manifests itself in two different ways. (a) At high intensities λ and short exposure times T the photographic detector registers fewer events for a given average number of photons per grain λT than when λ is somewhat smaller (but λT is fixed). This can be explained by postulating a dead time τ following each incident photon so that only those photons contribute to produce the necessary number R of input events required for developability of the grain which arrive at least τ units apart. (b) Low intensity reciprocity failure denotes the condition in which the response, for a fixed average input λT , is again lowered, this time when the intensity of radiation λ is low; that is, the photons arriving at a grain tend to be spaced far apart. By assuming the type of mechanism discussed in the case of the coincidence counter, this type of reciprocity failure can also be explained. The dead time counter models and coincidence time counter models postulated for photographic detectors can be identified with the times required for the photochemical creation of electron hole pairs and the recombination for such pairs respectively.

Apart from this application of the counter models, other possibilities suggest themselves. The use of a Type I and a simple coincidence counter in series enables one to construct a device for detecting the presence of a pulse pair of known spacing embedded in a Poisson noise stream. This combination of counters suggests itself from range determination problems in space systems. Another possibility would be to use these counters in parallel. If the Poisson stream consists of a variety of particles of different energy E , and the coincidence counter can be constructed so that particles of energy $E < E_0$ produce only one transition ($S_0 \rightarrow S_1$) whereas particles with energy $E \geq E_0$ produce two or more, then the coincidence counter will respond primarily to the high energy particles. A dead time counter, on the other hand, will not differentiate between particles of different energy. By varying the level E_0 it is thus possible to estimate the energy distribution of the Poisson stream. Since energy is directly proportional to frequency what we have thus described is the discrete counterpart of a spectrum analyzer. This analysis may be useful in the study of separation of energy levels for high energy particles.

3. Methods

The method for obtaining estimates of the Poisson stream parameter is the following. We take as our datum the quantity $M(t)$ which is the observed total number of output counts from the detector in time T .

From $M(T)$ we need an estimate, say $f[M(t)]$, of the expected number of counts, given λ , in time T , $E\{M_\lambda(T)\}$. Asymptotically, i.e., for large T , we know by the law of large numbers that $M(T)$ will

approach $E\{M_\lambda(T)\}$ provided the variance of $M(T)$ is finite. Now since $M(T)$ is an asymptotically unbiased estimate of $E\{M_\lambda(T)\}$ we choose $f[M(T)] = M(T)$. Thus $M(T)$ is our estimate of the expected number of output counts. The mathematical model will provide a relation between the parameter λ of the Poisson stream and the expected number of output counts in a fixed time of observation T , namely $r(\lambda) = E\{M_\lambda(T)\}$. We then obtain an estimate of λ , call it $\hat{\lambda}$, by equating $M(T) = r(\lambda)$ and solving for λ . This yields a unique, asymptotically unbiased estimate provided the expression $r(\lambda)$ is continuous and monotonic in λ . In the case of counters the expression for $E\{M_\lambda(T)\}$ is derived from the Laplace transform of the inter-arrival time random variable using results from renewal theory. In the case of the photographic detector the expression for $E\{M_\lambda(T)\}$ is derived by direct argument.

The ideal detector is one whose estimate of λ call it λ^* has the lowest possible variance in the class of unbiased detectors. When we are dealing with Poisson inputs we can achieve a variance whose lower bound is given by the Cramer-Rao Theorem. In this case we know that a uniform minimum variance estimator of λT , when the process is observed for a total time T , is $N(T)$, the total number of input events. This estimate has variance λT and hence the variance of the best estimate of λ , i.e., λ^* , is obtained from

$$(1.3.1) \quad \text{var}\{\lambda^*\} = \text{var}\left\{\frac{N(T)}{T}\right\} = \frac{1}{T^2} \text{var}\{N(T)\} = \frac{\lambda}{T}.$$

Having settled on a method for estimating λ and an upper bound on the precision (lower bound on variance) attainable from any detector

we can define a measure of efficiency which will allow us to compare different detectors.

We define the quantum efficiency of a detector as the ratio of the variance of the estimate λ^* to the variance of the estimate $\hat{\lambda}$ produced by the given detector. This efficiency is a function of both λ and the length of the time interval of observation T , (and other parameters depending on the model) although in many cases it will be independent of T .

We shall be concerned here with asymptotic results and we shall obtain the asymptotic quantum efficiency in the following way: We obtain asymptotic results for $E\{M_\lambda(T)\} = r(\lambda)$ and then obtain the asymptotic estimate $\hat{\lambda}$ by solving the equation $M(T) = r(\lambda)$. To obtain the asymptotic variance of this estimate we expand $M(T) = r(\hat{\lambda})$, in a Taylor Series about λ , the true value of the parameter,

$$(1.3.2) \quad M(T) = r(\hat{\lambda}) = r(\lambda) + \frac{dr(\lambda)}{d\lambda} (\lambda - \hat{\lambda}) + \dots$$

or

$$(1.3.3) \quad M(T) - r(\lambda) = \frac{d E\{M_\lambda(T)\}}{d\lambda} (\lambda - \hat{\lambda}) + \dots$$

Squaring both sides of (1.3.3) and taking expectations (ignoring higher-order terms) we find the variance of $M(T)$ is related to the variance of the estimate of λ by the equation

$$(1.3.4) \quad \text{var}\{M(T)\} = \left[\frac{dr(\lambda)}{d\lambda} \right]^2 \text{var}\{\hat{\lambda}\}$$

or

$$(1.3.5) \quad \text{var}\{\hat{\lambda}\} = \left[\frac{d E\{M_{\lambda}(T)\}}{d\lambda} \right]^{-2} \cdot \text{var}\{M(T)\} .$$

Since the ideal detector estimates λ by λ^* which we know has variance λ/T we note that the asymptotic efficiency, \mathcal{E} , is given by

$$(1.3.6) \quad \mathcal{E} = \frac{\text{var}\{\lambda^* \mid N(T)\}}{\text{var}\{\hat{\lambda} \mid M(T)\}}$$

or

$$(1.3.7) \quad \mathcal{E} = \frac{\lambda}{T} \cdot \frac{\left[\frac{d E\{M_{\lambda}(T)\}}{d\lambda} \right]^{-2}}{\text{var}\{M(T)\}} .$$

The discussion in this study is concerned largely with such devices as the radiation detectors and counting tubes used and studied by physicists, and for these the measure \mathcal{E} is extremely useful. Other measures of performance would need to be devised for detectors whose function it is to decide between alternative states of nature. For example, in the case of two such states, a criterion of efficiency might involve not only the probability of a correct decision, but the time involved in reaching a decision. This would be especially true in monitoring fallout, where the object is to decide whether the level is or is not critical. A simpler criterion for evaluating a detector which must decide which of two possible states prevails would be the total probability of error over a given time span. We shall evaluate a particularly interesting detector with respect to this criterion in Chapter V.

CHAPTER II

THE DEAD TIME COUNTERS

1. Results From Renewal Theory

We shall follow Feller (3) in deriving the Laplace transform of the inter-arrival time random variable for the Type I and Type II counters, since the type of argument used, particularly for the Type II counters, lends itself to deriving a corresponding result for a simple coincidence counter mechanism. However before deriving these results, we will look into some aspects of renewal theory which will be helpful in arriving at the efficiency of the detectors under consideration.

What we require is a relation between the mean and the variance of the inter-arrival distribution and the mean and variance of the number of events that will occur in a given time. These results are discussed by Smith (14), and an elementary treatment is given by Saaty (15). Let X_1 be the random variable which denotes the elapsed time between successive events (renewals), all of which, except perhaps the first, are assumed identically distributed. Then, we can write

$$(2.1.1) \quad T_n = X_1 + \dots + X_n$$

to denote the waiting time random variable up to, and including, the n^{th} event. This waiting time random variable has a distribution related to the distribution function $F(t)$ of X_1 , which is given by

$$(2.1.2) \quad F^{(n)}(t) = \int_0^t F^{(n-1)}(t-x) dF(x)$$

where $F^{(n)}(t)$ denotes the n -fold convolution of $F(t)$ with itself.

The X_1 are also assumed to be independently distributed.

Let exactly M events take place in a given time T . Label this $M(T)$. The probability of this happening is

$$(2.1.3) \quad P_M = F^{(M)}(T) - F^{(M+1)}(T)$$

since this is the probability that the M^{th} event but not the $(M+1)^{\text{st}}$ event takes place prior to T . From P_M we can then derive the mean and variance of the number of events in $(0, T)$. Letting $E\{M(T)\}$ and $\text{var}\{M(T)\}$ denote these parameters, and μ and σ^2 the mean and variance of the distribution $F(T)$, we have, from the key renewal theorem of Smith (19) that, as $T \rightarrow \infty$

$$(2.1.4) \quad E\{M(T)\} \approx \frac{T}{\mu}$$

and

$$(2.1.5) \quad \text{var}\{M(T)\} \approx \frac{\sigma^2 T}{\mu^3}.$$

These results can be viewed in the following non-rigorous manner. Heuristically we may write

$$(2.1.6) \quad E\{X_1 + \dots + X_{M(T)}\} = \mu E\{M(T)\},$$

that is, the expected time for M events should approximate the expected time for one event multiplied by the expected number of events. Likewise we may also write

$$\text{var}\{X_1 + \dots + X_{M(T)}\} = \sigma^2 E\{M(T)\}$$

remembering this is an approximation based on the same argument since

the variance of a sum of independent random variables is the sum of the variables. Thus we may also write in the same approximate vein

$$(2.1.7) \quad E[X_1 + \dots + X_{M(T)} - \mu M(T)]^2 = \sigma^2 E(M(T))$$

Furthermore, for large T we expect

$$(2.1.8) \quad X_1 + \dots + X_{M(T)} \approx T$$

The actual value is of the order $T - \mu$, (recall μ is the mean waiting time for an event and thus for the first event) which for large M can be approximated by T , hence using (2.1.6)

$$(2.1.9) \quad E(M(T)) \approx \frac{T}{\mu}$$

and substituting in (2.1.7) we get

$$(2.1.10) \quad E\{T - \mu M(T)\}^2 \approx \frac{\sigma^2 T}{\mu}$$

Multiply both sides by $1/\mu^2$ obtaining

$$(2.1.11) \quad E\{M(T) - \frac{T}{\mu}\}^2 \approx \frac{\sigma^2 T}{\mu^3}$$

and since $\frac{T}{\mu}$ can be replaced by $E\{M(T)\}$, we get

$$(2.1.12) \quad \text{var}\{M(T)\} \approx \frac{\sigma^2 T}{\mu^3}$$

With these results in mind, we return to the evaluation of counters.

2. The Type I Counter

The operation of a Type I counter can be described briefly as follows:

A counter which immediately registers the first input event and then only those input events, which are not preceded during a fixed time interval by a registered (output) event.

The results for the Type I counter are those which are most easily obtained. The dead time is assumed to be a constant τ and the counter operates on a Poisson process which has for inter-arrival distribution of input events the negative exponential distribution $1-e^{-\lambda t}$. Thus the inter-arrival time of registered events have a distribution (after occurrence of the first event)

$$(2.2.1) \quad F_0(t) = \begin{cases} 0 & t \leq \tau \\ 1-e^{-\lambda(t-\tau)} & t > \tau \end{cases}$$

$F_0(t)$ is used throughout to describe the output inter-arrival time distribution, $F(t)$ has been and will continue to be used for the input inter-arrival time distribution.

The Laplace transform of t is

$$(2.2.2) \quad \begin{aligned} \varphi_0(s) &= \int_0^{\infty} e^{-st} dF_0(t) \\ &= \int_0^{\infty} e^{-st} \lambda e^{-\lambda(t-\tau)} dt \\ &= \frac{\lambda}{\lambda+s} e^{-s\tau} \end{aligned}$$

Since

$$(2.2.3) \quad \mu_0 = \int_0^{\infty} t dF_0(t) = - \left. \frac{d \varphi_0(s)}{ds} \right|_{s=0}$$

and

$$(2.2.4) \quad \sigma_0^2 + \mu_0^2 = \int_0^\infty t^2 dF_0(t) = \left. \frac{d^2 \varphi_0(s)}{ds^2} \right|_{s=0}$$

we obtain the mean and variance of the inter-arrival distribution of counted events:

$$(2.2.5) \quad \mu_0 = \frac{\lambda\tau+1}{\lambda}$$

and

$$(2.2.6) \quad \sigma_0^2 = \frac{1}{\lambda^2}$$

Employing (2.1.4) and (2.1.12), we get

$$(2.2.7) \quad E\{M(T)\} = \frac{\lambda T}{\lambda\tau+1}$$

and

$$(2.2.8) \quad \text{var}\{M(T)\} = \frac{\lambda T}{(\lambda\tau+1)^3} .$$

Before we compute the quantum efficiency, it is interesting to note that for large values of λ , that is, strong incident radiation, we get

$$(2.2.9) \quad E\{M(T)\} \rightarrow T/\tau$$

and

$$(2.2.10) \quad \text{var}\{M(T)\} \rightarrow 0 .$$

This result can be viewed in the following way. In the limit as $\lambda \rightarrow \infty$ we get counts periodically and immediately following the end of the last

dead time, so that the number of counts becomes strictly proportional to the observation time, independent of λ , and thus the variance is zero.

The quantum efficiency is obtained from the expression (1.3.7)

$$(2.2.11) \quad \mathcal{E} = \frac{\lambda}{T} \left[\frac{dE(M(T))}{d\lambda} \right]^2 \frac{1}{\text{var}(M(T))}$$

which, in this case, turns out to be

$$(2.2.12) \quad \mathcal{E} = \frac{\lambda}{T} \frac{T^2}{(\lambda\tau+1)^4} \frac{(\lambda\tau+1)^3}{\lambda T}$$

or

$$(2.2.13) \quad \mathcal{E} = \frac{1}{\lambda\tau+1}.$$

We see that for non-zero τ the quantum efficiency tends monotonically to zero as $\lambda \rightarrow \infty$; that is, for strong radiation, the Type I counter is not very efficient as an estimator of λ . The number of counts increases with the time of observation, but tends, for large λ , to become independent of the radiation strength and only a function of the dead time. The efficiency depends on the arrival rate λ and is independent of the time of observation T or the total number of incident events.

3. The Type II Counter

The operation on the Type II counter can be stated as follows:

A counter which immediately registers those, and only those input events, which are not preceded during a fixed time interval by an input event.

We follow Feller's (3) argument in establishing the inter-arrival time distribution of output events for the Type II counter. It should be noted just from the physical situation that the number of registered events in the presence of strong incident radiation tends to zero for any given time interval T .

Let

$$(2.3.1) \quad p = e^{-\lambda\tau}$$

and

$$(2.3.2) \quad q = 1 - e^{-\lambda\tau}$$

denote respectively the probability of a non-arrival and an arrival of an incident event during a time span τ . Then the probability, once a counter is paralyzed, that v events will prolong the dead time is $q^v p$. Let X_1 denote the elapsed time between input event numbers $(i-1)$ and i . Then we can write

$$(2.3.3) \quad T(v) = X_1 + \dots + X_v + \tau$$

for the total time that the counter is paralyzed by exactly v events. This is a random variable, conditioned on v , which is the sum of v independent, identically distributed random variables. It follows that the Laplace transform of $T(v)$ is the product of $e^{-s\tau}$ and the v^{th} power of the Laplace transform of X_1 . Once we have the transform of $T(v)$, we can multiply $q^v p$ and sum over v to obtain the Laplace transform of the unconditional paralyzed time random variable which we need to evaluate the Type II detector.

The probability that an event will occur in a time $(0, t)$ with $t \leq \tau$ given that it occurs in a time interval $(0, \tau)$ is

$$(2.3.4) \quad \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda \tau}} = \frac{1 - e^{-\lambda t}}{q}$$

and this is the conditional probability $P(X_1 \leq t)$, for $i=1, \dots, v$ and gives rise to the Laplace transform

$$(2.3.5) \quad \int_0^\tau e^{-st} \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda \tau}} dt = \frac{1}{q} \cdot \frac{\lambda}{\lambda + s} [1 - e^{-(\lambda + s)\tau}]$$

so that the Laplace transform of $T(v)$ is simply

$$(2.3.6) \quad \varphi_v(s) = \frac{1}{q} \left(\frac{\lambda}{\lambda + s} \right)^v [1 - e^{-(\lambda + s)\tau}]^v e^{-s\tau}.$$

The unconditional Laplace transform is obtained by summing

$$(2.3.7) \quad \sum_{v=0}^{\infty} e^{-s\tau} \cdot \frac{1}{q} \left\{ \frac{\lambda}{\lambda + s} [1 - e^{-(\lambda + s)\tau}] \right\}^v q^v p$$

which yields

$$(2.3.8) \quad \frac{(\lambda + s)e^{-(s + \lambda)\tau}}{s + \lambda e^{-(s + \lambda)\tau}}.$$

This is the transform of the paralyzed time distribution. To obtain the transform of the inter-arrival time of registered events, we must take into account the additional time from the end of paralysis to the arrival of another incident event. The latter has Laplace transform $\frac{\lambda}{\lambda + s}$ so that we obtain finally the Laplace transform of the inter-arrival time distribution of registered events as

$$\begin{aligned}
 \varphi_0(s) &= \frac{1}{\lambda+s} \cdot \frac{(\lambda+s) e^{-(s+\lambda)\tau}}{s+\lambda e^{-(s+\lambda)\tau}} \\
 (2.3.9) \quad &= \frac{\lambda e^{-(s+\lambda)\tau}}{s + \lambda e^{-(s+\lambda)\tau}} .
 \end{aligned}$$

The next step is, as before, to obtain the mean and variance of the inter-arrival time distribution. It turns out, both in this case as well as others, that it is much less tedious to obtain the derivatives of $\frac{1}{\varphi(s)}$ rather than $\varphi(s)$. Noting that

$$(2.3.10) \quad \frac{d}{ds} \left[\frac{1}{\varphi(s)} \right] = -\varphi'(s) \left[\frac{1}{\varphi(s)} \right]^2$$

and that $\varphi(0) = 1$ we have

$$(2.3.11) \quad \mu = \varphi'(0) = \frac{d}{ds} \left[\frac{1}{\varphi(0)} \right] .$$

Also

$$(2.3.12) \quad \frac{d^2}{ds^2} \left[\frac{1}{\varphi(s)} \right] = -\varphi''(s) \left[\frac{1}{\varphi(s)} \right]^2 + \frac{2[\varphi'(s)]^2}{[\varphi(s)]^3}$$

so that since

$$(2.3.13) \quad \mu^2 + \sigma^2 = \frac{d^2}{ds^2} [\varphi(0)]$$

we have

$$(2.3.14) \quad \sigma^2 = -\frac{d^2}{ds^2} \left[\frac{1}{\varphi(0)} \right] + \left\{ \frac{d}{ds} \left[\frac{1}{\varphi(0)} \right] \right\}^2 .$$

For the Type II counter we have

$$(2.3.15) \quad \frac{1}{\varphi_0(s)} = \frac{s}{\lambda} e^{(\lambda+s)\tau_{+1}}$$

so that

$$(2.3.16) \quad \frac{d}{ds} \left[\frac{1}{\varphi_0(s)} \right] = \frac{1}{\lambda} e^{(\lambda+s)\tau} + \frac{s\tau}{\lambda} e^{(\lambda+s)\tau}$$

and

$$(2.3.17) \quad \frac{d^2}{ds^2} \left[\frac{1}{\varphi_0(s)} \right] = \frac{2\tau}{\lambda} e^{(\lambda+s)\tau} + \frac{s\tau^2}{\lambda} e^{(\lambda+s)\tau} .$$

Consequently

$$(2.3.18) \quad \frac{d}{ds} \left[\frac{1}{\varphi_0(0)} \right] = \frac{1}{\lambda} e^{\lambda\tau}$$

and

$$(2.3.19) \quad \frac{d^2}{ds^2} \left[\frac{1}{\varphi_0(0)} \right] = \frac{2\tau}{\lambda} e^{\lambda\tau} .$$

Hence

$$(2.3.20) \quad \mu_0 = \frac{e^{\lambda\tau}}{\lambda}$$

and

$$(2.3.21) \quad \sigma_0^2 = \frac{e^{2\lambda\tau}}{\lambda^2} - \frac{2\tau e^{\lambda\tau}}{\lambda} .$$

Again using (2.1.4) and (2.1.12) we obtain the mean and variance of the number of counts in an interval $[0, T]$. These are

$$(2.3.22) \quad E\{M(T)\} = \lambda T e^{-\lambda\tau}$$

and

$$(2.3.23) \quad \text{var}\{M(T)\} = \lambda T e^{-2\lambda\tau} [e^{\lambda\tau} - 2\tau\tau] .$$

From these quantities we proceed to deduce the efficiency of the Type II counter. Using (2.2.11), we obtain readily

$$(2.3.24) \quad \epsilon = \frac{\lambda}{T} \frac{(Te^{-\lambda\tau} - \tau\lambda Te^{-\lambda\tau})^2}{\lambda T(e^{\lambda\tau} - 2\lambda\tau) e^{-2\lambda\tau}}$$

$$\epsilon = \frac{(1 - \lambda\tau)^2}{(e^{\lambda\tau} - 2\lambda\tau)} .$$

The character of this result suggests that we look at the Type II counter a little more carefully. Formula (2.3.24) asserts that the efficiency of this type of counter is unity when λ is zero, decreases monotonically to zero at $\lambda = \frac{1}{\tau}$ and then becomes positive again although it tends to zero for $\lambda = \infty$.

This type of behavior is not quite so surprising if we take another look at the distribution of the output counts, $M(T)$. From equation (2.3.22) we see that the response function $r(\lambda) = E\{M_\lambda(T)\}$ is not monotonic in λ . As the radiation strength increases, the expected number of counts increases only until $\lambda = \frac{1}{\tau}$. For $\lambda > \frac{1}{\tau}$ the expected number of counts actually decreases with increasing λ , since the amount of time that the counter is paralyzed now becomes very large. A consequence of this is that the function

$$(2.3.25) \quad r(\lambda) = \lambda T e^{-\lambda\tau}$$

has no unique inverse. The only way to infer a unique value of λ from a single observation $M(T)$ is to have additional information about λ which guarantees that either λ is less than $\frac{1}{\tau}$ or λ is larger than $\frac{1}{\tau}$. In the absence of such information it is necessary to have an auxiliary experiment. We could, for example introduce an

absorption device into the Poisson stream which reduces its intensity by a known factor Q . The reduced stream of intensity $Q\lambda$ will, if $\lambda < \frac{1}{T}$, produce a reduced expected number of counts in time T . If $\lambda > \frac{1}{T}$ and Q is near unity we would tend to get a larger number of output events from the reduced stream. This auxiliary experiment would thus allow us to estimate, uniquely, the parameter λ of the Poisson stream. We see from this that it should not usually prove to difficult to obtain a unique estimate of λ from two observations of $M(T)$.

CHAPTER III

COINCIDENCE COUNTERS

1. Introduction

A number of different problems in a variety of contexts can be constructed, all of which have at their core the existence of an integer valued random process, $X_j(t)$, which, starting from the state S_0 strives to reach some preassigned state S_R . The process can increase by integer amounts, but if no jump occurs for a sufficiently long period of time, the process will revert to its initial or ground state, and must begin its climb towards state S_R all over again.

We can imagine a psychologist constructing a complex learning or maze experiment in which "success" depends on the performance of R successive tasks, and if any task is not performed with sufficient alacrity, the subject must begin from scratch. Such a process also occurs in the study of photographic detectors, in the study of Lasers, and, in fact in a great variety of scientific and engineering situations.

The fact that neither Type I nor Type II counters (and presumably intermediate types) are very good detectors for large values of λ would lead one to suspect that in those cases where an extreme range of parameter values λ is a priori likely to occur, different counters ought to be employed. As the discussion of coincidence counters will show, this type of device should prove of some value in such cases and moreover it provides a new model for various physical situations such as those mentioned above.

2. The Simple Coincidence Counter

As model for a simple coincidence counter, we shall consider the following mechanism: An incoming event impinges upon the counter which is so constructed as to be able to hold this event in memory for a time τ (a fixed constant). If another event occurs during the time that the first event is in memory, the counter registers an output event and resets its memory to zero content. If no second event occurs during time τ , the first event is lost, that is, produces no count.

The formal definition of a simple coincidence counter can be stated as:

A counter which immediately registers an output count whenever an input event is preceded during a fixed time interval τ by a nonregistered input event.

Such a situation may be obtained, for example, on a photographic emulsion in which the first photon incident on a silver halide grain creates an electron-hole pair which has a lifetime τ before recombination takes place. If, however, another pair is formed shortly after the first, a latent image speck, that is, a developable silver nucleus, or, at any rate, a stable, sublatent image speck (i.e., a stable speck but one too small to be developed) may be formed. Another conceivable situation in which a coincidence counter model might be useful is that of light amplification by stimulated emission of radiation - the so-called LASER. In this device, photons are required to be stored until a sufficient number are accumulated which are then released simultaneously to produce an intense monochromatic, coherent light pulse. If an insufficient number are captured during the holding

time of a ruby crystal or other lasing material the photons which arrive during this time do not produce a light pulse and are wasted.

It should be noted that the coincidence model suggested above differs from the general counter model proposed by Takacs in that the occurrence of an event within τ units of an earlier one produces a regeneration point (in effect, a new origin) for the output stochastic process in our case, but not in Takacs' case. In his process, an output event could occur after the arrival of both the second and the third input event. In our model, that would be considered physically impossible. To get two output counts, at least four input events are required.

The Laplace transform of the output inter-arrival time random variable is derived as follows: Beginning at a time when an output event has just occurred, let X_0, X_1, X_2, \dots denote the time between further successive input events. Assuming that the N^{th} event produces the first count, we have $X_1 > \tau, X_2 > \tau, \dots, X_{N-1} > \tau$, but $X_N \leq \tau$ (note that there are no conditions on X_0). Let $T_{(N)} = X_0 + X_1 + \dots + X_N$ denote the total time between output events. Then the conditional Laplace transform of $T(N)$, given N , is

$$(3.2.1) \quad E[e^{-sT(N)} | N] = E[e^{-sX_0}] E[e^{-s \sum_{i=1}^{N-1} X_i} | X_i > \tau] E[e^{-sX_N} | X_N \leq \tau].$$

Noting that the conditional probability that two input events are X units apart, given that they are not more than τ units apart, has density

$$(3.2.2) \quad \frac{\lambda e^{-\lambda X}}{1 - e^{-\lambda \tau}}$$

we find the conditional Laplace transform (i.e., the transform of this conditioned random variable) is

$$(3.2.3) \quad E[e^{-sX} | X \leq \tau] = \int_0^{\tau} e^{-sX} \frac{\lambda e^{-\lambda X}}{1 - e^{-\lambda \tau}} dX = \frac{\lambda(1 - e^{-(s+\lambda)\tau})}{(s+\lambda)(1 - e^{-\lambda \tau})}.$$

Similarly, if we know that two input events differ in time by at least τ units, we obtain the conditional Laplace transform

$$(3.2.4) \quad E[e^{-sX} | X > \tau] = \int_{\tau}^{\infty} e^{-sX} \frac{\lambda e^{-\lambda X}}{e^{-\lambda \tau}} dX = \frac{\lambda e^{-(s+\lambda)\tau}}{(s+\lambda)e^{-\lambda \tau}}.$$

From these results we obtain the conditional Laplace transform of $T(N)$ given N

$$(3.2.5) \quad \begin{aligned} E[e^{-sT} | N] &= E[e^{-sX_0}] \{E[e^{-sX} | X > \tau]\}^{N-1} E[e^{-sX} | X \leq \tau] \\ &= \frac{\lambda}{\lambda + s} \left\{ \frac{\lambda e^{-(s+\lambda)\tau}}{(s+\lambda)e^{-\lambda \tau}} \right\}^{N-1} \frac{\lambda(1 - e^{-(s+\lambda)\tau})}{(s+\lambda)(1 - e^{-\lambda \tau})} \\ &= \left(\frac{\lambda}{\lambda + s} \right)^{N+1} \frac{e^{-(N-1)(s+\lambda)\tau}}{e^{-(N-1)\lambda \tau}} \cdot \frac{1 - e^{-(s+\lambda)\tau}}{1 - e^{-\lambda \tau}}. \end{aligned}$$

To obtain the unconditional transform of $T(N)$ we multiply by the probability that the first output event will occur at the N^{th} input event, namely, $(1 - e^{-\lambda \tau})(e^{-\lambda \tau})^{N-1}$ and sum over N obtaining

$$(3.2.6) \quad E[e^{-sT}] = \sum_{N=1}^{\infty} E[e^{-sT} | N] (1 - e^{-\lambda \tau})(e^{-\lambda \tau})^{N-1}.$$

This yields

$$(3.2.7) \quad E[e^{-sT}] = \left(\frac{\lambda}{\lambda + s} \right)^2 (1 - e^{-(\lambda + s)\tau}) \sum_{N=1}^{\infty} \left[\frac{\lambda e^{-(s+\lambda)\tau}}{s + \lambda} \right]^{N-1}$$

and after summing the infinite series

$$(3.2.8) \quad \varphi_0(s) = \left(\frac{\lambda}{\lambda+s}\right)^2 \frac{1 - e^{-(\lambda+s)\tau}}{1 - \frac{\lambda}{\lambda+s} e^{-(\lambda+s)\tau}}.$$

In this formula, the term $\lambda^2/(\lambda+s)^2$ represents the transform of the waiting time for the second input event of the Poisson stream. As $\tau \rightarrow \infty$ the second input event can be taken as producing an output event, which is what we would expect of an infinite memory device.

Proceeding in a fashion similar to that used for the Type II counters, we obtain after some manipulation,

$$(3.2.9) \quad \frac{1}{\varphi_0(s)} = \frac{\lambda+s}{\lambda} \left[\frac{s}{\lambda(1 - e^{-(\lambda+s)\tau})} + 1 \right]$$

$$(3.2.10) \quad \frac{d}{ds} \left[\frac{1}{\varphi_0(s)} \right] = \frac{1}{\lambda} \left[\frac{s}{\lambda(1 - e^{-(\lambda+s)\tau})} + 1 \right]$$

$$+ \frac{\lambda+s}{\lambda} \left[\frac{1}{\lambda(1 - e^{-(s+\lambda)\tau})} - \frac{s\tau}{\lambda} \frac{e^{-(\lambda+s)\tau}}{(1 - e^{-(\lambda+s)\tau})^2} \right]$$

$$\frac{d^2}{ds^2} \left[\frac{1}{\varphi_0(s)} \right] = \frac{1}{\lambda} \left[\frac{2}{\lambda(1 - e^{-(\lambda+s)\tau})} \right]$$

$$(3.2.11) \quad + \frac{\lambda+s}{\lambda} \left[-\frac{\tau}{\lambda} \frac{e^{-(s+\lambda)\tau}}{(1 - e^{-(\tau+s)\tau})^2} \right]$$

$$+ s [\dots]$$

so that

$$(3.2.12) \quad \mu_0 = \frac{1}{\lambda} + \frac{1}{\lambda(1 - e^{-\lambda\tau})} = \frac{2 - e^{-\lambda\tau}}{\lambda(1 - e^{-\lambda\tau})}$$

$$(3.2.13) \quad \sigma_0^2 = \left[\frac{2-e^{-\lambda\tau}}{\lambda(1-e^{-\lambda\tau})} \right]^2 - \frac{2}{\lambda^2(1-e^{-\lambda\tau})} + \frac{\tau}{\lambda} \frac{e^{-\lambda\tau}}{(1-e^{-\lambda\tau})^2}.$$

Now

$$(3.2.14) \quad E(M_\lambda(T)) \approx \frac{\lambda T(1-e^{-\lambda\tau})}{2-e^{-\lambda\tau}}$$

$$(3.2.15) \quad \text{var}(M_\lambda(T)) \approx \frac{\lambda T(1-e^{-\lambda\tau})}{(2-e^{-\lambda\tau})^3} [\lambda\tau e^{-\lambda\tau} - 2(1-e^{-\lambda\tau}) + (2-e^{-\lambda\tau})^2].$$

Let us now differentiate $E(M_\lambda(T))$ with respect to λ

$$(3.2.16) \quad \begin{aligned} \frac{d}{d\lambda} E(M_\lambda(T)) &= T(1-e^{-\lambda\tau})(2-e^{-\lambda\tau})^{-1} \\ &+ \tau\lambda T e^{-\lambda\tau} (2-e^{-\lambda\tau})^{-1} - \tau\lambda T e^{-\lambda\tau} (1-e^{-\lambda\tau})(2-e^{-\lambda\tau})^{-2}. \end{aligned}$$

Hence the efficiency of the coincidence counter is

$$(3.2.17) \quad \begin{aligned} \epsilon &= \frac{\lambda}{T} (T(1-e^{-\lambda\tau})(2-e^{-\lambda\tau})^{-1} + \tau\lambda T e^{-\lambda\tau} (2-e^{-\lambda\tau})^{-1} \\ &- \tau\lambda T e^{-\lambda\tau} (1-e^{-\lambda\tau})(2-e^{-\lambda\tau})^{-2})^2 \times \\ &\left(\frac{(2-e^{-\lambda\tau})^3}{\lambda T(1-e^{-\lambda\tau}) [\lambda\tau e^{-\lambda\tau} - 2(1-e^{-\lambda\tau}) + (2-e^{-\lambda\tau})^2]} \right) \end{aligned}$$

or

$$(3.2.18) \quad \epsilon = \frac{[(1-e^{-\lambda\tau})(2-e^{-\lambda\tau})^{-1} + \lambda\tau e^{-\lambda\tau} (2-e^{-\lambda\tau})^{-1} - \lambda\tau (1-e^{-\lambda\tau})(2-e^{-\lambda\tau})^{-2} e^{-\lambda\tau}]^2}{(1-e^{-\lambda\tau})^2 (2-e^{-\lambda\tau}) [\lambda\tau e^{-\lambda\tau} - 2(1-e^{-\lambda\tau}) + (2-e^{-\lambda\tau})^2]}$$

Such a coincidence counter resembles a Type I counter in that the expected number of counts $E(M_\lambda(T))$ is monotonically increasing in λ . It differs from this type of counter in that the expected number of counts is asymptotically proportional to the number of input events as τ becomes large. This is also reflected in its quantum efficiency which, although more complex in form, tends to increase from zero for $\lambda = 0$ to 100 percent as λ tends to infinity.

The response function, $r(\lambda)$, of the coincidence counter is clearly monotonic since formula (3.2.16) which is its derivative has no zeros in $(0, \infty)$. Also, as is apparent from the expression for ξ , the efficiency of the counter in estimating the parameter of a Poisson stream is independent of T .

In actual practice a coincidence counter could be expected also to have a short dead time after registered events, so that a cross between a Type I and a pure coincidence counter would have to be considered.

It is also interesting to speculate as to the suitability of such a counter for separating a periodic signal from contaminating Poisson sources. If, for example, a coincidence counter with variable time constant, τ , could be constructed and a periodic pulse with repetition rate λ_0 were incident as well as a Poisson source with mean rate λ , we would have that

$$(3.2.19) \quad E(M(T)) \approx \lambda T$$

for τ a little less than $\frac{1}{\lambda_0}$ and $\lambda \ll \lambda_0$ (since, in combination with an adjacent periodic pulse every random pulse produces a count) but

$$(3.2.20) \quad E(M(T)) = \frac{(\lambda + \lambda_0)T}{2}$$

for $\tau \geq \frac{1}{\lambda_0}$, so that a discontinuity in the plot of $E(M_\lambda(T))$ as a function of τ would indicate the presence of the periodic pulse train if present. (For $\tau \ll \lambda_0$ we would expect that periodicity would be irrelevant so that in that case

$$E(M(T)) = \frac{(\lambda + \lambda_0) E(1 - e^{-(\lambda + \lambda_0)\tau})}{2 - e^{-(\lambda + \lambda_0)\tau}}$$

as in (3.2.14)).

Coincidence counters have been constructed and used (15) to study radiation which does not have Poisson character. Here again, the plot of $E(M_\lambda(T))$ for fixed mean input rate λ and fixed observation time T , but variable time constant τ should be useful as a comparison with one obtained from a known source. One can use it to infer the existence of non-randomness in the unknown source as well as to estimate the parameters of interest.

3. The R-Fold Coincidence Counter

One natural extension of the simple coincidence counter is to permit input events arriving within specified time intervals to change state levels of a counter, and allow the registering of an output event only after a terminal state, say after the R^{th} input event, is reached. We term this the R-fold coincidence counter. For $R = 2$, this reduces to the previously described simple coincidence counter which has two states S_0 and S_1 . Then an input event changes the counter from S_0 to S_1 , and if another event occurs before time τ , an output count is

registered and the counter immediately reverts to S_0 . The counter also immediately reverts to S_0 at time τ without registering if the input does not arrive within the time interval.

Briefly we can describe the R-fold coincidence counter as follows:

A counter which immediately registers an output event whenever an input event is preceded by (R-1) non-registered input events none of which are spaced more than τ units apart and τ is a time interval of random length whose (R-1) values are independently and identically distributed.

Here we shall derive the waiting time distribution (or, rather, its Laplace transform) for the first time occurrence of the critical level R in the case where the waiting time between upward jumps of unit magnitude has negative exponential distribution, $F(t) = 1 - e^{-\lambda t}$. This is equivalent to the assumption that input impulses constitute a Poisson process. We also assume that all the levels S_j , $j=1,2,\dots,R-1$ can decay only to S_0 and that such decay takes place at the end of an interval of duration τ_j following the latest incident impulses. We assume that each τ_j is itself a random variable with absolutely continuous density, and with distribution function $G(\tau)$ which is independent of S_j and the times of occurrence of the states S_j . The special case $R = 2$; $G(\tau) = 0$ for $\tau < \tau_0$, $G(\tau) = 1$ for $\tau \geq \tau_0$, was treated by another method in the previous section.

We shall not attempt to evaluate the efficiency of such a general coincidence counter explicitly since the mathematical manipulations become too tedious and unmanageable. Ad hoc cases of special interest can be handled by computers. Our concern in this section is primarily

to arrive at the formula for the inter-arrival time distribution from which, by the methods covered in earlier sections, the efficiency can be derived for those specific cases that may be of interest.

The process with which we are concerned is non-Markovian in the sense that merely knowing which state S_j , $j=1,2,\dots,R-1$, the process is in at a given time t is not sufficient to calculate the probable behavior of the process beyond this time. In addition, we need to know how long ago the last impulse occurred. The only case in which such knowledge is not needed is when at time t the process is discovered in state S_0 .

We shall write $p_j(t,\tau)$ for the probability density that jointly: a) at time t the process is in state S_j , b) that it reached this state τ time units ago, that is, at $t - \tau$; c) that the state at $t = 0$ is S_0 , and that previous to time t the state R was not reached. Now the probability that a decay to S_0 takes place during the small interval $(\tau, \tau + \delta)$ after occurrence of the last impulse is $\delta G'(\tau)$, and the probability that no decay takes place during the time interval $(t-\tau, t)$ is $1-G(\tau)$. Hence the conditional probability of decay during the interval $(\tau, \tau + \delta)$, given that no decay has taken place during the elapsed time τ since the last impulse, label it $\delta\mu(\tau)$, is given by

$$(3.3.1) \quad \delta\mu(\tau) = \frac{\delta \cdot G'(\tau)}{1 - G(\tau)}.$$

Later on, it will be useful to consider the solution of this differential equation, hence we note here that from the above one can obtain

$$(3.3.2) \quad G(\tau) = 1 - e^{-\int_0^\tau \mu(x) dx}$$

The function $\mu(\tau)$ is sometimes called the hazard function, and has been discussed, for example, in connection with telephone call demands and other queuing problems. In the particular case where $G(\tau)$ is a negative exponential distribution, it is seen that $\mu(\tau)$ is constant, so that the conditional probability of decay after τ time units from the last impulse have elapsed is independent of τ . It is in that sense that the negative exponential distribution is said to have no memory.

We shall derive here a set of differential equations from which the Laplace transform inter-arrival time distribution of output counts can be obtained. First let us obtain the probability that the state of the process will be S_j ($j > 0$) at time $(t+\delta)$ and that $\tau+\delta$ time units have elapsed since the last impulse, given that at time t the time elapsed since the last impulse was τ and write it as a function of $p_j(t, \tau)$. We assume here that, for small δ , the probability of an impulse during a time interval δ is $\lambda\delta$ and the probability of no impulse is $1-\lambda\delta$; and that other cases have such small probability that they can be neglected. These are the usual assumptions linked to the fact that the input impulses form a Poisson stream. In terms of $p_j(t, \tau)$ and the hazard function $\mu(\tau)$, we can write

$$(3.3.3) \quad p_j(t+\delta, \tau+\delta) = (1-\lambda\delta)(1-\delta\mu(\tau))p_j(t, \tau)$$

or, neglecting terms in δ^2 since δ is assumed small we get

$$(3.3.4) \quad p_j(t+\delta, \tau+\delta) = [1-\lambda\delta-\delta\mu(\tau)]p_j(t, \tau).$$

If during the interval $(t, t+\delta)$ an impulse occurs, we have, for the probability that the state is S_j at $t+\delta$ and that the time t' since the last impulse was less than δ [denote this by $p_j(t+\delta, t' < \delta)$]

$$(3.3.5) \quad p_j(t+\delta, t' < \delta) = \lambda\delta \int_0^\infty (1-\delta\mu(\tau))p_{j-1}(t, \tau)d\tau$$

where the integrand on the right represents the probability that the state was S_{j-1} at time t , had been attained τ time units earlier; and did not spontaneously decay in the time span $(\tau, \tau+\delta)$ following the moment it was attained. Since all possible times, τ , of attainment of the state S_{j-1} must be considered, the probability of no decay from state S_{j-1} during $(t, t+\delta)$ is $\int_0^\infty (1-\delta\mu(\tau))p_{j-1}(t, \tau)d\tau$. Again neglecting terms in δ^2 this becomes

$$(3.3.6) \quad p_j(t+\delta, \tau' < \delta) = \lambda\delta \int_0^\infty p_{j-1}(t, \tau)d\tau.$$

Note that $\int_0^\infty p_{j-1}(t, \tau)d\tau$ is the probability that the state is S_{j-1} at time t regardless of how long ago the state was attained. For use later on we note that we can approximate $p_j(t, \delta, \tau' < \delta)$ by $\delta \cdot p_j(t, \delta, \tau' = 0)$ since the time interval δ is small, so that the probability of an impulse at some instance τ' in $(0, \delta)$ is nearly the same for all τ' in this interval [see (3.3.20)].

These considerations hold for $j=1,2,\dots,R-1$. For state S_0 , we have

$$(3.3.7) \quad p_0(t+\delta) = (1-\lambda\delta)p_0(t) + \sum_{j=1}^{R-1} (1-\lambda\delta) \int_0^\infty \delta \mu(\tau) p_j(t,\tau) d\tau$$

i.e., the probability of being in state S_0 at time $t+\delta$ is the probability of being in that state at time t and having no incident events in $(t,t+\delta)$ plus the probability that a decay occurs in the period $(t,t+\delta)$, when the process is in state S_j ($j=1,2,\dots,R-1$) at time t and no incident events occur during $(t,t+\delta)$. Note that the integral under the summation represents the probability of a decay in the period $(t,t+\delta)$ when the process is in state S_j ($j=1,2,\dots,R-1$) at time t and this probability is independent of how long ago the state S_j was attained previous to time t . Similarly for $p_0(t+\delta, \tau' < \delta)$ we have by (3.3.5)

$$(3.3.8) \quad p_0(t+\delta, \tau' < \delta) = \lambda\delta \int_0^\infty (1-\delta\mu(\tau)) p_{R-1}(t,\tau) d\tau.$$

In the period $(t,t+\delta)$ a transition from state S_{R-1} to state S_R may occur thus causing the process to revert immediately to state S_0 and register an output count, or we may write

$$(3.3.9) \quad p_0(t+\delta) = (1-\lambda\delta)p_0(t) + \sum_{j=1}^{R-1} \int_0^\infty \lambda\mu(\tau) p_j(t,\tau) d\tau$$

and

$$(3.3.10) \quad p_0(t+\delta, \tau' < \delta) = \int_0^\infty p_0(t+\delta, \delta') d\delta' = \lambda\delta \int_0^\infty p_{R-1}(t,\tau) d\tau$$

if terms in δ^2 are neglected. If in (3.3.4) we transpose $p_j(t, \tau)$, then divide each side of the equation by δ and pass to the limit, we obtain

$$\lim_{\delta \rightarrow 0} \frac{p_j(t+\delta, \tau+\delta) - p_j(\tau, t)}{\delta} = \lim_{\delta \rightarrow 0} - \frac{(\lambda\delta + \delta\mu(\tau))}{\delta} p_j(t, \tau)$$

or

$$(3.3.11) \quad \frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial \tau} = -(\lambda + \mu(\tau)) p_j(t, \tau) .$$

This is so for $j=1, 2, \dots, R-1$ and similarly from (3.3.9) we get for $j = 0$

$$(3.3.12) \quad \frac{dp_0}{dt} = -\lambda p_0(t) + \sum_{j=1}^{R-1} \int_0^{\infty} \mu(\tau) p_j(t, \tau) d\tau .$$

These are the differential equations whose solutions will provide the inter-arrival time distribution of the output counts. First we will obtain an expression for the probability of attaining state R at some point in $[0, T]$, $\int_0^T p_R(t') dt'$. From this we find the probability density for the waiting time until state R is reached. This is obtained by differentiating $\int_0^T p_R(t') dt'$ with respect to T , since this can be interpreted as the probability that the waiting time for reaching state R will not exceed T . The remainder of this discussion is therefore concerned with obtaining this distribution by developing its Laplace transform.

Equation (3.3.11), treated as a differential equation in the variable τ , can be solved by taking the Laplace transform with

respect to the variable t . Label $\pi_j(s, \tau)$ the Laplace transform of $p_j(t, \tau)$ or

$$(3.3.13) \quad \pi_j(s, \tau) = \int_0^{\infty} e^{-st} p_j(t, \tau) dt .$$

Now take the Laplace transform of each side of (3.3.11) and (3.3.12) and note that

$$(3.3.14) \quad \int_0^{\infty} e^{-st} \frac{dp_0}{dt} dt = -p_0(0) + s \pi_0(s)$$

with $p_0(0) = 1$, while

$$(3.3.15) \quad \int_0^{\infty} e^{-st} \frac{\partial p_j}{\partial t} dt = -p_j(0, \tau) + s \pi_j(s, \tau) = s \pi_j(s, \tau)$$

with $p_0(0, \tau) = 0$ for $j \neq 0$.

Hence equation (3.3.11) becomes

$$(3.3.16) \quad s \pi_j(s, \tau) + \frac{\partial}{\partial \tau} \pi_j(s, \tau) = -[\lambda + \mu(\tau)] \pi_j(s, \tau)$$

and thus by rewriting we get

$$(3.3.17) \quad \pi_j(s, \tau) = \pi_j(s, 0) e^{-(s+\lambda)\tau} e^{-\int_0^{\tau} \mu(x) dx}$$

or, using (3.3.2)

$$(3.3.18) \quad \pi_j(s, \tau) = \pi_j(s, 0) e^{-(s+\lambda)\tau} [1-G(\tau)] .$$

Since we are concerned only with the marginal distribution of $p_j(t, \tau)$, we integrate over τ to obtain [setting $1-G(\tau) = g(\tau)$] and

using a star, *, to denote the Laplace transform]

$$(3.3.19) \quad \int_0^{\infty} \pi_j(s, \tau) d\tau = \pi_j(s, 0) \int_0^{\infty} e^{-(s+\lambda)\tau} q(\tau) d\tau = \pi_j(s, 0) q^*(s+\lambda) .$$

From equation (3.3.5), we obtain in the limit as is already indicated in the discussion following that equation

$$(3.3.20) \quad p_j(t, 0) = \lambda \int_0^{\infty} p_{j-1}(t, \tau) d\tau$$

and, taking the Laplace transform,

$$(3.3.21) \quad \pi_j(s, 0) = \lambda \int_0^{\infty} \pi_{j-1}(s, \tau) d\tau$$

for $j = 1, 2, \dots, R$. Since transitions from the state S_0 do not depend on τ , we have

$$(3.3.22) \quad \pi(s, 0) = \lambda \pi_0(s) .$$

Using (3.3.19), (3.3.21), and (3.3.22), we obtain

$$(3.3.23) \quad \pi_j(s, 0) = \{q^*(s+\lambda)\}^{j-1} \lambda^j \pi_0(s) .$$

From (3.2.18), we obtain

$$(3.3.24) \quad \int_0^{\infty} \pi_j(s, \tau) \mu(\tau) d\tau = \pi_j(s, 0) \int_0^{\infty} e^{-(s+\lambda)\tau} q(\tau) \mu(\tau) d\tau$$

and using (3.3.1), this becomes

$$(3.3.25) \quad \int_0^{\infty} \pi_j(s, \tau) \mu(\tau) d\tau = \pi_j(s, 0) \int_0^{\infty} e^{-(s+\lambda)\tau} G'(\tau) d\tau \\ = \pi_j(s, 0) G'^*(s+\lambda).$$

Finally, we take the Laplace transform of (3.3.12).

$$(3.3.26) \quad -1 + s \pi_0(s) = -\lambda \pi_0(s) + \sum_{j=1}^{R-1} \int_0^{\infty} \pi_j(s, \tau) \mu(\tau) d\tau$$

and substituting the result (3.3.25),

$$(3.3.27) \quad -1 + s \pi_0(s) = -\lambda \pi_0(s) + \sum_{j=1}^{R-1} \pi_j(s, 0) G'^*(s+\lambda) \\ = -\lambda \pi_0(s) + \pi_0(s) G'^*(s+\lambda) \times \sum_{j=1}^{R-1} \{g^*(s+\lambda)\}^{j-1} \lambda^j.$$

On summing we obtain the result

$$(3.3.28) \quad -1 + s \pi_0(s) = -\lambda \pi_0(s) + G'^*(s+\lambda) \lambda \pi_0(s) \left[\frac{1 - \{\lambda g^*(s+\lambda)\}^{R-1}}{1 - \lambda g^*(s+\lambda)} \right]$$

or,

$$(3.3.29) \quad \pi_0(s) = \frac{1}{s + \lambda - \lambda G'^*(s+\lambda) \left[\frac{1 - \{\lambda g^*(s+\lambda)\}^{R-1}}{1 - \lambda g^*(s+\lambda)} \right]}.$$

If we let $H(t)$ denote the probability that a count (that is, state S_R) is attained in the interval $(0, T)$, we have from the fact that

$$(3.3.30) \quad H(T) = \int_0^T p_R(t') dt'$$

or, on differentiating,

$$(3.3.31) \quad H'(T) = p_R(T)$$

that

$$(3.3.32) \quad E[e^{-st} R] = [H'(T)]^* = \pi_R(s)$$

is the desired Laplace transform of the waiting time distribution towards which we have been working. For state S_R , we also have

$$(3.3.33) \quad \pi_R(s) = \pi_R(s, 0)$$

since the system immediately returns to state S_0 . Hence, using (3.3.21),

$$(3.3.34) \quad \pi_R(s) = \lambda \int_0^\infty \pi_{R-1}(s, \tau) d\tau$$

and substituting from (3.3.19),

$$(3.3.35) \quad \pi_R(s) = \lambda \pi_{R-1}(s, 0) \}^*(s+\lambda)$$

or, iterating this substitution as in (3.3.23),

$$(3.3.36) \quad \pi_R(s) = \lambda^R \{q^*(s+\lambda)\}^{R-1} \pi_0(s) .$$

Thus,

$$(3.3.37) \quad E[e^{-st} R] = \frac{\lambda^R (g^*(s+\lambda))^{R-1}}{s + \lambda - \lambda G^*(s+\lambda) \left[\frac{1 - (\lambda g^*(s+\lambda))^{R-1}}{1 - \lambda g^*(s+\lambda)} \right]}$$

is the Laplace transform of the inter-arrival arrival distribution of output counts from the general R-fold Laplace transform coincidence counter.

For $R = 2$; $G(\tau) = 0$, for $\tau < \tau_0$ and $G(\tau) = 1$ for $\tau \geq \tau_0$ which is the case treated previously in Chapter 3, section 2 we have

$$(3.3.38) \quad G^*(s+\lambda) = \int_{\tau_0}^{\infty} e^{-(s+\lambda)\tau} d\tau = e^{-(s+\lambda)\tau_0}$$

and

$$(3.3.39) \quad E[e^{-st} R_2] = \left(\frac{\lambda}{\lambda+s}\right)^2 \frac{1 - e^{-(s+\lambda)\tau_0}}{1 - \left(\frac{\lambda}{\lambda+s}\right) e^{-(s+\lambda)\tau_0}}$$

which is the result previously derived for this case by the conditional probability approach used in that section.

Another interesting case which is mathematically tractable is that in which G is a negative exponential distribution, $G(\tau) = 1 - e^{-\xi\tau}$. In this case we find

$$(3.3.40) \quad G^*(s+\lambda) = \int_0^{\infty} e^{-(s+\lambda)\tau} \xi e^{-\xi\tau} d\tau = \frac{\xi}{s+\lambda+\xi}$$

and

$$(3.3.41) \quad E[e^{-st} R_2] = \frac{\lambda^2 s^2 + 2\lambda^3 s + \lambda^2 \xi s + \lambda^3 \xi + \lambda^4 - \lambda^2 \xi}{s^4 + (4\lambda + \xi)s^3 + (6\lambda^2 + 4\lambda\xi)s^2 + (4\lambda^3 + 5\lambda^2\xi)s + 2\xi\lambda^3}$$

CHAPTER IV

QUANTUM EFFICIENCY OF PHOTOGRAPHIC DETECTORS

1. On the Problem of Specifying a Measure of Performance

In one sense, the perfect photographic detector is one in which only a single incident photon is required to make a grain developable. Consequently, if an average of λT photons is incident per grain during the time of exposure, T , and if the Poisson distribution applied, the probability of a grain being made developable is the sum of the probabilities of 1, 2, 3, ... photons incident on a grain i.e.,

$$(4.1.1) \quad H_{\lambda}[0, T] = H(\lambda T) = \sum_{j=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} = 1 - e^{-\lambda T}.$$

An "imperfect" emulsion, however, can be constructed in at least three ways.

In the first place, the grains may not be tightly packed. In this case, even though the flux in photons per grain is λT and a fraction $H(\lambda T)$ of the grains in a tightly packed emulsion could be made developable, only a fraction $Q \cdot H(\lambda T)$ are actually made developable, where Q is the ratio of the number of grains present to the number that could be packed into the same surface area. If only a single photon is required for developability, the fractional number of grains made developable in such a case is

$$(4.1.2) \quad Q \cdot H(\lambda T) = Q \sum_{j=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} = Q(1 - e^{-\lambda T})$$

In the second place, the grains may not be good absorbers, so that the effective flux per grain is not λT but, say, $Q\lambda T$, where

Q now denotes the absorption of a grain. This is equivalent to assuming perfect absorption and placing a filter of transmission Q over the emulsion. Then the number of grains made developable (assuming one photon per grain required for developability) would be

$$(4.1.3) \quad H_Q(\lambda T) = \sum_{j=1}^{\infty} e^{-Q\lambda T} \frac{(Q\lambda T)^j}{j!} = 1 - e^{-Q\lambda T}.$$

However, it is also possible to think of the photographic detector as being imperfect in the sense that a grain might require $R = 1/Q$ photons to become developable. (In this case we can think of Q as a measure of the inertia of a grain.) If each grain were so constituted, the fraction made developable would be the sum of the probabilities of $R, R + 1, R + 2, \dots$ photons incident on a grain, i.e.,

$$(4.1.4) \quad \begin{aligned} H_R(\lambda T) &= \sum_{j=R}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} \\ &= 1 - e^{-\lambda T} [1 + \lambda T + \dots + \frac{(\lambda T)^{R-1}}{(R-1)!}] \end{aligned}$$

Formulas (4.1.2) - (4.1.4) indicate the three quite different ways in which a "fractional utilization factor, Q " could be introduced. In general, an emulsion will be imperfect as a result of all these reasons. For a combination of the last two reasons we could write

$$(4.1.5) \quad H_{R \cdot Q}(\lambda T) = \sum_{j=R}^{\infty} e^{-Q\lambda T} \frac{(Q\lambda T)^j}{j!}$$

or, even more generally,

$$(4.1.6) \quad H_{R_1, Q_1, Q}(\lambda T) = \sum_{i=1}^{\infty} Q_i \sum_{j=R_1}^{\infty} e^{-(Q\lambda T)} \frac{(Q\lambda T)^j}{j!}$$

in case the emulsion consists of a variety of grains and the fraction Q_1 of the population requires R_1 photons for developability. In formula (4.1.6), Q_1 represents the fractional area covered by grains of inertia R_1 when all the grains absorb a fraction Q of the incident energy. (Holes are considered as grains with infinity inertia.)

Aside from the fact that "fractional utilization" factor is an ambiguous notion as shown above, there are other reasons why the old definition of quantum efficiency as a ratio of number of output to input events (which is linked to the notion of fractional utilization) is not a good measure of performance.

If every input event gave rise to a single count we should like to say that the detector has a quantum efficiency of unity. Even if every input event gave rise to several counts we should still wish to speak of the device as having unit quantum efficiency, and we should consider each aggregate of counts produced by one input event as a single output event. Suppose, however that on the average, only every other input event is effective in producing output events and that for each output two counts are registered. The ratio of output to input counts is one and yet a detector in which every input event produces an output count may be viewed as superior. It follows that a different approach to a unique and meaningful definition of quantum efficiency is called for--one which cannot only be computed theoretically but can also be measured experimentally. This is accomplished by applying to photographic detectors the concepts developed in Chapter I.

2. The Quantum Efficiency of Simple "Photographic" λ -Detectors

We shall consider a (simple) photographic detector as an ensemble of go-no-go detectors, each of which is capable of responding just once immediately following the arrival of the R^{th} event. Consequently, further arrivals at a detector having already received R "hits" are wasted, and cannot be transferred to another detector or registered in the output. The photographic detector is unique in that it can operate simultaneously on a large set of Poisson processes, namely, on all those sources in space which are imaged on the face of the photographic plate. Also, because of imperfect imaging and scatter within the emulsion, a Poisson point source is imaged, not on one, but on a set S of detectors (or photographic grains). Thus, in comparing the intensity of two point sources, the output from $2S$ detectors must be compared. An ideal non-photographic ensemble of S detectors exposed to a Poisson source of intensity λ for a time T would provide an estimate $\hat{\lambda}$ of λ with a variance of $\frac{\lambda}{ST}$, assuming the set of S detectors to operate independently of each other on the same Poisson source.

If we suppose that each photographic detector requires exactly R hits to become developable and retains its developability indefinitely, then we can calculate that the mean number of detectors responding to an intensity λ (per grain per unit time) after an exposure time T will be

$$\begin{aligned}
 (4.2.1) \quad S \cdot H(\lambda T) &= S \left[\sum_{j=R}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} \right] \\
 &= S \left[1 - e^{-\lambda T} \sum_{j=0}^{R-1} \frac{(\lambda T)^j}{j!} \right] .
 \end{aligned}$$

Now if H is the probability that a detector will respond, and if S independent experiments are performed, the probability that exactly k responses are obtained is

$$(4.2.2) \quad p(k) = \binom{S}{k} H^k (1-H)^{S-k}$$

and hence the variance of the number of responses is

$$(4.2.3) \quad \sigma_S^2 = S \cdot H(\lambda T) [1-H(\lambda T)] .$$

Since (4.2.1) yields the response function $r(\lambda T)$ of the photographic detector (known in photographic theory as the characteristic curve) and (4.2.3) the variance of the output, we can use (1.3.5) to obtain the variance of the asymptotically unbiased estimate of λT , $r^{-1}(\lambda T)$, so that

$$(4.2.4) \quad \text{var}(\hat{\lambda} | M(T)) = \frac{H(1-H)}{S} \left[\frac{dH}{d(\lambda T)} \right]^{-2} .$$

As indicated earlier, the ratio of this variance to that of the ideal detector is a measure of the efficiency of the detecting process.

To illustrate the calculation of quantum efficiency with a simple example, suppose that we have a photographic detector requiring one photon hit to make each grain developable. If the total radiation on one grain during the time of exposure has average λT , the fractional number of grains which will become developable is

$$(4.2.5) \quad H(\lambda T) = \sum_{j=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} = 1 - e^{-\lambda T}.$$

If there are a total of S grains within a given area, we expect there to be

$$(4.2.6) \quad S[H(\lambda T)] = S(1 - e^{-\lambda T})$$

developed grains after exposure and processing. Due to random variations in the incident events, there will be fluctuations in the number of developed grains from one area of S grains to the next. The standard deviation of the number of developed grains is obtained from the binomial distribution and is $\sqrt{S(1 - e^{-\lambda T})e^{-\lambda T}}$ (using $1 - e^{-\lambda T}$ as the probability of success). The estimate of λ is

$$(4.2.7) \quad \hat{\lambda} = H^{-1}(\lambda T) = -\frac{1}{T} \log(1 - H(\lambda T))$$

and the variance of the estimate is by (1.3.5)

$$(4.2.8) \quad \begin{aligned} \text{var}(\hat{\lambda}) &= \text{var}\{S[H(\lambda T)]\} \left[\frac{dS[H(\lambda T)]}{d(\lambda T)} \right]^{-2} \\ &= \frac{S e^{-\lambda T} (1 - e^{-\lambda T})}{S^2 T^2 e^{-2\lambda T}} \end{aligned}$$

whereas the variance of $\hat{\lambda}$ in the case of a set of S ideal non-photographic detectors is

$$(4.2.9) \quad \text{var}(\hat{\lambda}) = \frac{\lambda}{ST}.$$

Thus the quantum efficiency is

$$\begin{aligned}
 \zeta &= \zeta(\lambda T, R=1) = \frac{\lambda}{8T} \frac{8T^2 e^{-2\lambda T}}{(1-e^{-\lambda T})e^{-\lambda T}} \\
 &= \frac{\lambda T e^{-\lambda T}}{1-e^{-\lambda T}},
 \end{aligned}
 \tag{4.2.10}$$

and this depends on the total expected number of input events λT rather than on the input rate λ . We also see that ζ is a monotonically decreasing function of λT , and that the one photon photographic detector has 100% efficiency only at $\lambda T = 0$. For given λ , this detector becomes increasingly inefficient as T increases. The photographic type of detector is a saturating device which has infinite memory.

In general the photographic detector requiring R photons for developability has its peak quantum efficiency at about $\lambda T = R-1$. This follows from the fact that $dH/d(\lambda T)$ is a Gamma function in λT with parameter $R-1$, and hence its mode is at $R-1$. It follows that if the background radiation is known to have mean value λT , the ideal photographic detector is the one which requires $R = \lambda T + 1$ photons for developability rather than one photon. At the value of $\lambda T = 1$, the efficiency of the R -photon detector has the form

$$\zeta = \frac{R-1}{H(1-H)} \left[\frac{e^{-(R-1)} (R-1)^{R-1}}{(R-1)!} \right]^2.
 \tag{4.2.11}$$

Using Stirling's formula for $n!$

$$n! \sim \sqrt{2\pi} n^{n + \frac{1}{2}} e^{-n}
 \tag{4.2.12}$$

we obtain

(4.2.13)

$$\epsilon \sim \frac{1}{2\pi H(1-H)} .$$

The maximum value of $H(1-H)$ is .25, so that a lower bound for the quantum efficiency peak value is

(4.2.14)

$$\epsilon_{\max} \approx \frac{2}{\pi} \approx 0.65$$

independent of the number of photons, R , required for developability. This is perhaps the most surprising result concerning this class of radiation detectors.

When the detection problem is one of detecting the smallest possible incremental signal for a given radiation background, the corresponding R -photon emulsion must be considered as the ideal photographic detector; the efficiency of an actual emulsion relative to this detector is obtained by dividing the maximum quantum efficiency of the actual emulsion by the quantum efficiency of the appropriate R -photon photographic detector. This yields a somewhat higher value of efficiency than appears in the literature.

However, even the higher value of efficiency obtained in this manner is not a really fair index of improvability for actual photographic materials. The reason for this is that the assumption of a given background radiation is unrealistic. We do not know the strength of the background radiation. This means that the photographic detector must have a certain latitude. The latitude we require is a measure of our a priori ignorance as to the strength of the radiation background. All we can do in the face of this ignorance is to make the quantum efficiency as uniform as possible for the range of background radiation strengths that we expect to encounter. If, for example, we expect a

range of 1 to 10 photons per grain area for the background radiation, we might take for our ideal photographic detector a composite of $R = 1, \dots, 8$. Such a composite detector made up, say of an equal number of grains of each type has a more uniform, but lower, quantum efficiency than a one-photon detector would have. If a still greater latitude is required, say for a range of λT 's from 1 to 100, a uniform quantum efficiency over this range can be achieved only by using a still greater range of R 's.

If we want to determine the optimum response curve over a given range, $(\lambda_0 T, \lambda_1 T)$ for uniform quantum efficiency throughout this range, we proceed as follows: Since

$$(4.2.15) \quad \zeta = \left[\frac{dH}{d(\lambda T)} \right]^2 \cdot \frac{\lambda T}{H(1-H)}$$

and since we require

$$(4.2.16) \quad \zeta = \text{constant} \quad \lambda_0 T < \lambda T < \lambda_1 T$$

we obtain

$$(4.2.17) \quad H'(\lambda T) = \zeta^{\frac{1}{2}} \{H(\lambda T)[1-H(\lambda T)]\}^{\frac{1}{2}} / (\lambda T)^{\frac{1}{2}}.$$

This leads to

$$(4.2.18) \quad H(\lambda T) = \sin^2 \left\{ \zeta^{\frac{1}{2}} [(\lambda T)^{\frac{1}{2}} - (\lambda_0 T)^{\frac{1}{2}}] \right\}$$

with $\lambda_0 T < \lambda T < [\pi/2 \zeta^{\frac{1}{2}} + (\lambda_0 T)^{\frac{1}{2}}]^2$ for the response curve having the optimum shape. With $\lambda_0 T = 1$ and a uniform quantum efficiency of 1% a value $\lambda_1 T \approx \frac{\pi^2}{4 \times .01} \approx .250$ is obtained. The range is thus 1 to 250.

Conversely, if the range is restricted to the values $\lambda_0 T = 1$, $\lambda_1 T = 100$, the greatest possible uniform quantum efficiency is found to be about 2.5%.

It follows from these considerations that actual emulsions, which have, at present, a peak quantum efficiency of between 0.5% and 1.0% , differ from the attainable optimum by a factor which is certainly less than 10. The improvability of photographic materials, given the constraints imposed by pictorial photography, is thus much smaller than the factor of 100 which was believed possible, and no further dramatic breakthrough in photographic "speed" should be anticipated.

3. The Photographic Detector with Reciprocity Failure

As in the previous section, we postulate that a photographic detector consists of an ensemble of go-no-go detectors. Once one of these detectors has acted (i.e., once a silver halide grain has become developable), further incident hits are wasted. Reciprocity failure means that the photographic detector responds not just to the total number of photons, λT , incident during the time of exposure, but reacts differently, depending on whether for $\lambda T = \text{constant}$ it is the time of exposure or the strength of radiation which is large. There are two types of reciprocity failure; high intensity failure, which can be attributed to a type of dead time phenomenon as in a Type I counter, and low intensity failure, which can be thought of as due to the finite memory of a coincidence type of counter mechanism. Diverse explanations of this phenomenon have been given (9) but the crucial experiments to determine the precise mechanism whereby reciprocity

failure is produced have not yet been attained. Consequently any model whose consequences are in reasonable accord with existing experimental data can provide a step forward.

We assume here that for photons in the visible region there is not sufficient energy to produce a developable grain. As is well known, at shorter wave lengths there is enough energy in a particle to trigger one or more photographic grains, and for such particles the problem of reciprocity failure does not arise.

For low intensity reciprocity failure, we assume that a photon can produce an imbalance in a silver halide crystal which can persist for a time τ . Another photon incident during this time will cause this imbalance to cease by producing an atomic silver speck. Such a speck may, or may not, in itself be developable. If it is, we have a two-photon photographic detector with low intensity reciprocity failure. If τ is infinite, this reduces to the type of photographic detector discussed in Section 2. It may be necessary to have a larger speck of silver to produce development than one obtained from two photons. If we assume that the two photon speck is non-developable but one twice as large is developable, then two further photon hits within an interval τ are required to produce either another speck or to enlarge the one already formed. Since actual emulsions are a mixture of grains of varying sensitivity, we would have to combine various models to simulate an actual photographic material. Here we shall content ourselves with some discussion of the two and four photon photographic detectors, as even these present considerable difficulty.

To arrive at the response curve and quantum efficiency of a two photon photographic detector with low intensity reciprocity failure, we should proceed as follows:

Beginning with the Laplace transform of the inter-arrival time distribution of the coincidence counter input (3.2.8), we find the inverse Laplace transform and integrate this transform from zero to T . The result, $F_0(\lambda, T, \tau)$, indicates the probability of one or more coincidence events which is the probability of a grain becoming developable during the time of exposure T , that is, the probability that at least one silver speck is formed in a grain irradiated by an average of λT photons, (and one such speck is sufficient for developability). If there are S photographic grains in the area under consideration, the expected number of grains which will contain a developable speck after exposure time T will then be $S \cdot F_0(\lambda, T, \tau)$. A plot of $F_0(\lambda, T, \tau)$ versus λ or T indicates the average fractional number of grains which become developable as λ or T increases, the other variable being held constant. (In photographic technology, when studying reciprocity failure, it is customary to hold $F_0(\lambda, T, \tau)$ (which corresponds to the developed optical density) constant and plot for various values of λ or T the value of λT needed to produce a fixed $F_0(\lambda, T, \tau)$).

Once we have $F_0(\lambda, T, \tau)$ we can obtain the variance in the number of developed grains from the binomial distribution, that is, the variance in the number of developable grains can be expected to be $S F_0(\lambda, T, \tau) \cdot (1 - F_0(\lambda, T, \tau))$. Since there are $S \lambda T$ photons expected during the time of exposure, an ideal detector would estimate λ with variance λ / ST so that the efficiency of the photographic detector is

$$(4.3.1) \quad \mathcal{E} = \frac{\lambda}{ST} \left[\frac{dSF(\lambda, T)}{d\lambda} \right]^2 \frac{1}{SF(\lambda, T)[1-F(\lambda, T)]}$$

If instead of two photons it takes four photons for developability, we must convolve F with itself to obtain the probability of at least two output events in the coincidence counter.

The problem of carrying out this procedure arises right at the start in trying to obtain an explicit closed form for the inverse Laplace transform. To find the inverse transform of the coincidence counter inter-arrival time

$$(4.3.2) \quad \varphi_0(s) = \left(\frac{\lambda}{\lambda+s} \right)^2 \frac{1 - e^{-(\lambda+s)\tau}}{1 - \frac{\lambda}{\lambda+s} e^{-(\lambda+s)\tau}}$$

one should take a contour integral over the left half plane. (From the fact that the distribution function is zero for $t < 0$, we know that all the poles of $\varphi(s)$ must have negative real part (9). Hence we must find the roots of $s + \lambda(1 - e^{-(\lambda+s)\tau})$.

To find these let

$$(4.3.3) \quad s = re^{i\theta} = r \cos \theta + ir \sin \theta$$

so that

$$(4.3.4) \quad e^{-\tau s} = e^{-\tau r \cos \theta} e^{-i\tau r \sin \theta}$$

and

$$(4.3.5) \quad e^{-i\tau r \cos \theta} = \cos(\tau r \cos \theta) - i \sin(\tau r \cos \theta)$$

so that

$$(4.3.6) \quad s + \lambda[1 - e^{-(s+\lambda)\tau}] = 0$$

becomes

$$(4.3.7) \quad \text{I. } r \cos \theta + \lambda - \lambda e^{-\tau \lambda - \tau r \cos \theta} \cos (\tau r \cos \theta) = 0$$

and

$$(4.3.8) \quad \text{II. } r \sin \theta - \lambda e^{-\lambda \tau} e^{-\tau r \cos \theta} \sin (\tau r \cos \theta) = 0 .$$

Squaring and adding these, we obtain

$$(4.3.9) \quad r = \pm \lambda [1 - e^{-\lambda \tau} e^{-\tau r \cos \theta}]$$

or

$$(4.3.10) \quad \cos \theta = - \frac{\lambda}{r} \pm \frac{1}{\tau r} \ln [1 \pm \frac{r}{\lambda}] .$$

Substituting this result in I above, we obtain after some reduction

$$(4.3.11) \quad \cos [\lambda \tau + \ln (\pm \frac{r}{\lambda})] = \frac{\ln (1 \pm \frac{r}{\lambda})}{\lambda \tau (1 \pm \frac{r}{\lambda})} .$$

A sketch of the expressions on either side of this equation reveals the locations of the infinity of roots. Some direct attempts at a solution by numerical methods indicate that large scale computer programming is necessary.

Thus a closed form approximation to the distribution function $F_0(\lambda, T, \tau)$ would be helpful. One possibility is to fit a Gamma distribution, i.e., a density function of the form

$$(4.3.12) \quad \Gamma(x; u, k) = \frac{u e^{-ux} (ux)^{k-1}}{(k-1)!}$$

by fitting the first few moments of this distribution to those obtainable by differentiating $\varphi_0(s)$. Since the first two moments of $\Gamma(x; uk)$ are

(4.3.13)

$$\mu_0 = \frac{k}{u}, \quad \sigma_0^2 = \frac{k}{u^2}$$

and these determine the particular Gamma distribution completely, we could set

(4.3.14)

$$\left[\frac{1}{\phi_0(s)}\right]'_{s=0} = \frac{k}{u}$$

and

(4.3.15)

$$\left[\left(\frac{1}{\phi_0(s)}\right)'\right]_{s=0}^2 - \left[\frac{1}{\phi_0(s)}\right]''_{s=0} = \frac{k}{u^2}.$$

If τ is large $\phi_0(s)$ tends to $\lambda^2/(\lambda+s)^2$, that is, the Laplace transform of a Gamma distribution with $k = 2$ and $\lambda = u$. This is as it should be, for that corresponds to the waiting time distribution for the second input event in the case of a Poisson process. For large enough values of τ , we are justified in approximating $F_0(\lambda, T, \tau)$ near the origin by a Gamma distribution with $k = 2$. (Some trial approximations easily show that this parameter is not very sensitive to variations in k .) Thus to a crude approximation we will need only u and this parameter is found from

(4.3.16)

$$\frac{2}{u} = \frac{1}{\lambda} + \frac{1}{\lambda(1-e^{-\lambda\tau})}$$

or

(4.3.17)

$$u = \frac{2\lambda(1-e^{-\lambda\tau})}{2-e^{-\lambda\tau}}$$

so that we shall take

$$(4.3.18) \quad F_0(\lambda, T, \tau) = \int_0^T \left[\frac{2\lambda(1-e^{-\lambda\tau})}{2-e^{-\lambda\tau}} \right]^2 t e^{-\left[\frac{2\lambda(1-e^{-\lambda\tau})}{2-e^{-\lambda\tau}} \right] t} dt$$

as response function in the case that two photons within time τ can produce developability. We shall take

$$(4.3.19) \quad F_0^{(2)}(\lambda, T, \tau) = \int_0^T \left[\frac{2\lambda(1-e^{-\lambda\tau})}{2-e^{-\lambda\tau}} \right]^4 \frac{t^3}{3!} e^{-\left[\frac{2\lambda(1-e^{-\lambda\tau})}{2-e^{-\lambda\tau}} \right] t} dt$$

1/2 twice as large a silver speck is required.

Formula (4.3.18) can be integrated to yield

$$(4.3.20) \quad F_0(\lambda, T, \tau) = 1 - e^{-uT} - uT e^{-uT}$$

which indicates that the response function of the two photon photographic detector with low intensity reciprocity failure is identical in shape (to this crude approximation) to that of the two photon detector without reciprocity failure but stretched along the λ -axis by the factor λ/u . Again to this approximation the efficiency of this detector is related to that of the two photon simple photographic detector by the relation

$$(4.3.21) \quad \mathcal{E}(\lambda, T, \tau) = \mathcal{E}(u) \left(\frac{du}{d\lambda} \right)^2$$

where $\mathcal{E}(u)$ is the efficiency of the simple two photon photographic detector evaluated at the abscissa value at which its response curve has the magnitude $F_0(\lambda, T, \tau)$. The term $\left(\frac{du}{d\lambda} \right)^2$ arises from the fact that $\mathcal{E}(u)$ involves the expression $\left(\frac{dF_0}{du} \right)^2$ whereas we need $\left(\frac{dF_0}{d\lambda} \right)^2$ in the expression for $\mathcal{E}(\lambda, T, \tau)$.

High intensity reciprocity failure can be handled in much the same way as low intensity reciprocity failure. In this case we postulate a dead time, τ , so that only those photons contribute to making an R-photon photographic detector respond which arrive at least τ units apart. In this case the probability that a grain will become developable in time T is obtained by calculating the probability of R or more output events from a Type I counter. Here again we encounter the apparent difficulty that as a first step we need the inverse Laplace transform of the R^{th} power of the inter-arrival time transform of the Type I counter

$$(4.3.22) \quad [\varphi_0(s)]^R = \left[\frac{\lambda}{\lambda+s} e^{-\tau s} \right]^R$$

which represents the waiting time distribution for the R^{th} output event from a Type I counter. Subsequently we need the integral from 0 to T which is the probability that the R^{th} event will occur prior to time T and corresponds to the probability that a grain exhibiting high intensity reciprocity failure will become developable.

However, in the present case, we can make use of a device that will also be found to be useful in a problem to be discussed in the following chapter. We replace T by $T' = T - (R-1)\tau$, a contracted time interval. It is easily shown that the output process from a Type I counter is again a Poisson Process with parameter λ in contracted time. Hence high intensity reciprocity failure is equivalent, under the present model, to shortening of the exposure time for an R photon simple photographic detector by an amount $(R-1)\tau$ --the accumulated dead time arising from the first $(R-1)$ input photons which are

incident on a photographic grain. Hence the efficiency of an R-photon photographic emulsion with high intensity reciprocity failure and dead time τ_d is found from

$$(4.3.23) \quad H(\lambda, T, \tau_d, R) = H(\lambda T', R)$$

and using

$$(4.3.24) \quad \frac{dH}{d\lambda}(\lambda, T, \tau_d, R) = \frac{dH(\lambda T', R)}{d\lambda}$$

we get

$$(4.3.25) \quad \xi(\lambda, T, \tau_d, R) = \frac{\lambda}{T} \cdot \frac{1}{H(\lambda, T, \tau_d, R)[1-H(\lambda, T, \tau_d, R)]} \left[\frac{dH(\lambda, T, \tau_d, R)}{d\lambda} \right]^2$$

which becomes

$$(4.3.26) \quad \xi(\lambda, T, \tau_d, R) = \frac{T'}{T} \cdot \xi(\lambda, T', R)$$

as can be seen by multiplying the numerator and denominator of

(4.3.25) by $T' = [T - (R-1)\tau_d]$ and using (4.3.23) and (4.3.24).

CHAPTER V

COMBINATIONS OF COUNTERS ACTING AS DETECTORS

So far we have been entirely concerned with detectors which are employed to estimate the parameter λ of a Poisson type input. Here we shall give some indication of what can be accomplished in other situations. First we consider a series arrangement of a dead time and a simple coincidence counter which is used as a detector for discovering an event consisting of two pulses with given spacing embedded in a Poisson process of noise pulses. What is of interest in such a case is the overall probability of failure, i.e., the probability of not sensing the twin signal pulses and the probability of mistaking two noise pulses for the twin signal pulse. This can arise in several physical situations, for example, in a radar range determination problem, and the discussion will proceed from that point of view. The second situation in this chapter is devoted to a detection situation in which a dead time and a coincidence counter are used in parallel. This combination suggests itself in the situation where the Poisson stream to be observed consists of events which have a variety of energy levels and it is desired to estimate the intensity of that portion of the stream whose energy exceeds a given threshold. This situation is the discrete counter part of a spectrum analyzer.

1. Dead Time and Coincidence Counters in Series (Radar Range Determination Problem)

A problem which can be solved, at least approximately, by considering a series combination of dead time and coincidence counters acting on a Poisson stream, is the following.

A space vehicle approaching the moon is required to fire its retro rocket at some predetermined distance from the moon. To sense this distance, pulse radar is used in the following manner. The radar uses two range gates. As soon as a pulse is received in the first gate the instrument ceases to respond for a time τ_d and then fires the retro rocket provided a signal pulse is received in the second range gate during the time interval τ_c following τ_d . If no pulse is received in this interval, the first pulse is judged to be spurious and the radar reverts to searching for the first range indication. This system can fail to operate if (a) a spurious pulse occurs just prior to the time at which the first turn indication would occur, thus immobilizing and preventing it from being in the proper state to detect the first range signal and (b) two spurious pulses occur earlier than the intended firing time spaced in such a way as to produce a pulse in both range gates and thus fire the retro rocket too soon. We are interested in determining the probability of both of these types of failures $[p(a) \text{ and } p(b)]$ as a function of the mean time between noise pulses, $1/\lambda$, and the total time of operation of the radar system, T .

The probability $p(a)$ is very simply obtained and is just the probability of a spurious event during the time interval $[T-\tau_d, T]$. Since the spurious events will be assumed to arrive in a Poisson stream with parameter λ , the probability of no events during this time interval is $1-e^{-\lambda\tau_d}$. The probability $p(b)$ is more difficult to obtain, but an approximate value can be obtained by the following line of reasoning.

The system described is equivalent to passing the Poisson stream first through a Type I counter with dead time τ_d , then through a hypothetical time contractor which reduces the total elapsed time by the amount of dead time $M\tau_d$ experienced due to the number, M , of output counts from the Type I counter which have occurred during the elapsed time. This results in a new Poisson stream on the interval $T' = T - M\tau_d$ with parameter

$$(5.1.1) \quad \lambda' = \frac{E(M_T)}{E(T')} .$$

This stream is fed into a coincidence counter with coincidence time constant τ_c and the probability of one or more coincidences during a time span T' , given that M output pulses are obtained from the first counter, can then be obtained by integrating the inverse Laplace transform of the inter-arrival time distribution. We then multiply this probability by the probability of getting M output pulses from the Type I counter and sum over M to obtain the probability $p(b)$.

This procedure is far from simple, however, and an approximation to $p(b)$ can be obtained by using the expected number of counts from the Type I counter

$$(5.1.2) \quad E(M(T)) = \frac{\lambda T}{\lambda \tau_d + 1}$$

and computing the probability of a coincidence for a Poisson process with parameter λ' over the expected time-contracted interval

$E(T') = T - E(M_T)\tau_d$. It is interesting to note that

$$(5.1.3) \quad \lambda' = \frac{E(M_T)}{E(T')}$$

becomes

$$(5.1.4) \quad \frac{\frac{\lambda T}{\lambda \tau_d + 1}}{T - \frac{\lambda T}{\lambda \tau_d + 1} \tau_d} = \lambda ,$$

that is, the output stream from the Type I counter in the contracted time interval T' is a stream with the same parameter as the input process.

Furthermore, this stream is a Poisson stream since the waiting time distribution from the termination of dead-time (which corresponds, in contracted time, to the arrival of an input event) to the next input event is a negative exponential distribution.

Now the expected number of counts $E\{M'\}$ in a coincidence counter over a time span T' is

$$(5.1.5) \quad E\{M'(T')\} = \frac{\lambda T' (1 - e^{-\lambda \tau_c})}{2 - e^{-\lambda \tau_c}}$$

which for $\lambda \tau_c \ll 1$ becomes

$$(5.1.6) \quad E\{M(T')\} = \frac{\lambda T' [1 - (1 - \lambda \tau_c)]}{2 - (1 - \lambda \tau_c)} \approx \lambda^2 \tau_c T' .$$

Let $P_{T'}(r)$ denote the probability of exactly r coincidences in the time T' . Since the expected number of coincidences is

$$(5.1.7) \quad E\{M(T')\} = \sum_{r=0}^{\infty} r P_{T'}(r)$$

and in the present situation $E\{M(T')\} \ll 1$ we may make the further approximation $P_{T'}(r) = 0$ for $r \geq 2$ so that

$$(5.1.8) \quad E\{M(T')\} = P_T(1) = \sum_{r=1}^{\infty} P_{T'}(r) .$$

The expression on the right is $p(b)$, the probability of one or more of the kind of coincidences which cause premature firing. For this approximation, therefore, we have

$$(5.1.9) \quad p(b) = \lambda^2 \tau_c T' .$$

As an example, let $\lambda = 10^{-2}$ counts per second and $T = 100$ seconds, $\tau_d = \tau_c = 0.5$ seconds then

$$(5.1.10) \quad E\{M\} = \frac{10^{-2} \times 10^2}{5 \times 10^{-3} + 1} \approx 1$$

hence $E\{T'\} = T - E\{M\} \tau_d \approx T$ (note that without the coincidence counter we would expect a noise pulse and hence premature firing.) Since $E\{M(T')\} = \lambda^2 \tau_c T = 5 \times 10^{-3}$, the probability of an early firing is only about 1/2 percent, and since $p(a) = 1 - e^{-\lambda \tau_d} = 5 \times 10^{-3}$ the overall probability of malfunction due to the incident Poisson noise is $p(a) + p(b) = .01$.

2. Coincidence and Dead Time Counters in Parallel - Energy Detection

It was remarked earlier that radiation can be considered as made up of discrete photons and that differences in the frequency or wavelength of the radiation can be associated with differences in the energy which the photons may have. In some applications a detector is required to measure not only the total number of photons in a stream but also the fraction having a given energy level. The continuous

analog of this type of a detector would be a spectrum analyzer which measures the relative power in a narrow band of frequencies.

Here we shall confine ourselves to a Poisson stream in which the photons can have two energy levels E_1 and E_2 . We shall further assume that the differences between these types of photons manifests itself in their ability to produce electrons in some counter device; the lower energy photons (E_1) being able to dislodge only one electron while the higher energy photons are capable of dislodging at least two electrons. The counters, whether dead time or coincidence time, are conceived of as operating on the electron stream produced by the photons. We are interested in observing both the intensity of the stream and the fractional intensity of the high energy radiations.

Assuming that all the electrons produced by a photon are generated within a time τ after the occurrence of the photon we can proceed as follows:

A type I dead time counter with time constant τ is used to observe the total intensity, $\lambda = \lambda_1 + \lambda_2$, of the overall Poisson stream. With this choice of time constant, the multiple events produced by the highly energetic photons will be reduced to simple counts and the formulas of Chapter II section 2 can be used to determine the efficiency as far as the estimation of λ is concerned.

In addition, a simple coincidence counter is introduced into the stream, with coincidence time constant τ . This counter will count the high energy photons (one count per high energy photon since the electrons are all produced within a short time τ of the occurrence of the photon) and will also register a count when two low energy photons occur within τ time units.

Let $n_1(T)$ and $n_2(T)$ denote the number of low and high energy photons occurring during an observation time T . An ideal device would in this case, produce output counts m^* which, for the overall λ -detector would be $m_d^* = n_1 + n_2$ and for the high energy, or λ_2 -detector, $m_c^* = n_2$. The non-ideal counts described above would yield outputs m_d and m_c related to those by

$$(5.2.1) \quad m_d \leq m_d^* = n_1 + n_2$$

and

$$(5.2.2) \quad m_c \geq m_c^* = n_2.$$

Letting $V^*(T)$ denote the ideal variance of the estimate of $\lambda_2/\lambda_1 + \lambda_2$ and $V(T)$ denote the variance of the estimate obtained from the combination of dead time and coincidence time counters, we can, as in the case of the photographic detector, define a measure of detector efficiency by

$$(5.2.3) \quad \xi = V^*(T)/V(T).$$

The variance of the ideal estimate of $\frac{\lambda_2}{\lambda_1 + \lambda_2}$ can be obtained from the distributions of the ideal estimates of λ_2 and $\lambda_1 + \lambda_2$.

We have

$$(5.2.4) \quad p(\hat{\lambda}_2 = \frac{r}{T}) = e^{-\lambda_2 T} \frac{(\lambda_2 T)^r}{r!}$$

and

$$(5.2.5) \quad p(\widehat{\lambda_1 + \lambda_2} = \frac{r+s}{T}) = e^{-(\lambda_1 + \lambda_2)T} \frac{(\lambda_1 + \lambda_2)^{r+s}}{(r+s)!}$$

or since the two kinds of Poisson events are assumed to be independent

$$(5.2.6) \quad p(\lambda_2 = \frac{r}{T}, \widehat{\lambda_1 + \lambda_2} = \frac{r+s}{T}) = e^{-\lambda_2 T} \frac{(\lambda_2 T)^r}{r!} e^{-\lambda_1 T} \frac{(\lambda_1 T)^s}{s!}.$$

The distribution of the ratio $\xi = \frac{\widehat{\lambda_2}}{\widehat{\lambda_1 + \lambda_2}}$ is consequently

$$(5.2.7) \quad p(\xi = \frac{r}{r+s}) = \sum_{k=1}^{\infty} p(\lambda_2 = kr) p(\lambda_1 = ks) \\ = \sum_{k=1}^{\infty} e^{-(\lambda_2 T)} \frac{(\lambda_2 T)^{kr}}{(kr)!} e^{-(\lambda_1 T)} \frac{(\lambda_1 T)^{ks}}{(ks)!}.$$

From this formula we obtain the variance of the ideal estimate $V^*(T)$ by means of

$$(5.2.8) \quad V^*(T) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \left(\frac{r}{r+s}\right)^2 \sum_{k=1}^{\infty} e^{-(\lambda_2 T)} \frac{(\lambda_2 T)^{kr}}{(kr)!} e^{-(\lambda_1 T)} \frac{(\lambda_1 T)^{ks}}{(ks)!} \\ - \left\{ \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{r}{r+s} \sum_{k=1}^{\infty} e^{-\lambda_2 T} \frac{(\lambda_2 T)^{rk}}{(kr)!} e^{-(\lambda_1 T)} \frac{(\lambda_1 T)^{ks}}{(ks)!} \right\}^2$$

We have not yet specified how the actual estimate of $\lambda_2/\lambda_1 + \lambda_2$ is to be obtained much less what its variance will be.

To obtain an estimate of λ_2 once we have obtained an estimate of $\lambda = \lambda_1 + \lambda_2$ we can proceed as follows:

We know that the expected number of counts from the coincidence counter will be

$$(5.2.9) \quad E[m_c] = E[n_2] + f(\lambda - \lambda_2, T)$$

where, using (3.1.14),

$$(5.2.10) \quad f(\lambda - \lambda_2, T) = \frac{(\lambda - \lambda_2)T[1 - e^{-(\lambda - \lambda_2)\tau}]}{2 - e^{-(\lambda - \lambda_2)\tau}}.$$

Using the observed number of counts m_c in place of $E(m_c)$, $\hat{\lambda}_2 T$ in place of $E(n_2)$ and $\hat{\lambda}$ in place of λ we obtain an expression

$$(5.2.11) \quad m_c = \hat{\lambda}_2 T + \frac{(\hat{\lambda} - \hat{\lambda}_2)T[1 - e^{-(\hat{\lambda} - \hat{\lambda}_2)\tau}]}{2 - e^{-(\hat{\lambda} - \hat{\lambda}_2)\tau}}$$

which can be solved for the estimate $\hat{\lambda}_2$. Hence also we obtain an estimate for the relative intensity $\hat{\lambda}_2/\hat{\lambda}$, since $\hat{\lambda}$ was obtained by (2.2.7).

In principle we could then obtain the variance of this estimate and the efficiency of this energy detector. The character of the expressions makes it clear that this involves extensive numerical work and perhaps Monte Carlo simulation.

In principle such a combination of counters can be used as the discrete analog of a spectrum analyzer. Where more than two energy levels are involved the coincidence counter would need to be "tuned" so that only those photons with an energy in excess of a given threshold will produce more than one electron. By varying this threshold, E , the proportion of photons having energy in excess of E could then be observed.

It should be noted that photographic detectors have, to some extent, the characteristics described above. High energy photons, for example X-rays, produce one or more developable grains, since they release a

number of electrons in passing through a photographic emulsion. Low energy photons contribute to developability but are required in greater numbers and over a limited time span in order to produce a developable grain.

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